بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

"In the name of Allah ,the Most Beneficent ,the Most Merciful."

شروع کوم په نامه د الله(ج) چې رحمٰن او رحیم دی.

Narrated Abu Hurairah (رضي الله عنه), the Prophet (صلى الله عليه وسلم), said, "None of you should observe Saum (fast) for a day or two ahead of Ramadan unless he has the habit of observing Saum (Nawafil) (and if his fasting coincides with that day) then he can observeSaum (fast) on that day." (Sahih)

[H 1914 Al-Bukhari, Arabic-English, Volume 3, By Dr Mohammad Muhsin Khan)]

ګرانو او قدرمنو وروڼو اوخویندو! السلام علیکم ورحمت الله وبرکاته!

له الله(ج) څخه هیله کوم چې تاسې ټول روغ او جوړ وئ٠ دغه لاندې د ریاضي درسونه به ان شاءالله ځینې پلرونو او په خاصه توګه، د دوی اولادونو ته ډېر ګټور وي٠

نوټ: دغه د درسونو لړۍ به ان شاءالله، له لومړي ټولګي څخه تر پوهنتونه پورې دوام ولري٠ د یادونې وړ ده، چې زما مسلک ریاضي نده، او ما خپلې زده کړې، د دکتورا تر کچې د الکترانکس په برخه کې، په انګلستان کې سرته رسولې٠ همدارنګه، ما ۶ کاله د کابل پوهنتون د انجینرۍ په پوهنځی، او ۲۷ کاله د انګلستان په یوه پوهنتون کې معلمي کړېده٠ یعنې هغه چا چې په الکترانکس انجینرنګ کې زده کړې کړې وي، نو هغه د ریاضي په معنٰی ډېر ښه پوهیږي٠ زه به ان شاءالله د ریاضي له درسونو وروسته، د الکترانکس په برخه کې درسونه شروع کړم٠

نوټ: زه د ریاضي د درسونو له ټولو قدرمنو لوستونکو څخه هیله کوم، چې له دغو درسونو څخه اعظمي ګټه پورته کړي. زه د خپل وخت ډېره قیمتي برخه ددغه درسونو په برابرولو کې تېروم. زه هغه وخت ډېر خوشحاله کېږم، چې یو ګران وطنوال له دغو درسونو ګټه پورته کړي. والسلام عبدالله وردک

Dear Brothers and Sisters! السلام علیکم و رحمت الله و برکاته

I kindly request each one of you to remember me, my parents, my children, my family and the entire Muslims of the world in your daily Prayers.

ګرانو او قدرمنو وروڼو اوخویندو! السلام علیکم ورحمت الله وبرکاته!

له الله(ج) څخه هیله کوم چې تاسې ټول روغ او جوړ وئ، او الله(ج) د وکړي، چې ماته ، زما مور او پلار ته، زما اولادونو ته ، زما ټولې کورنۍ ته او د نړۍ ټولو مسلمانانو ته ، د زړه له کومې دعا وکړئ٠ والسلام

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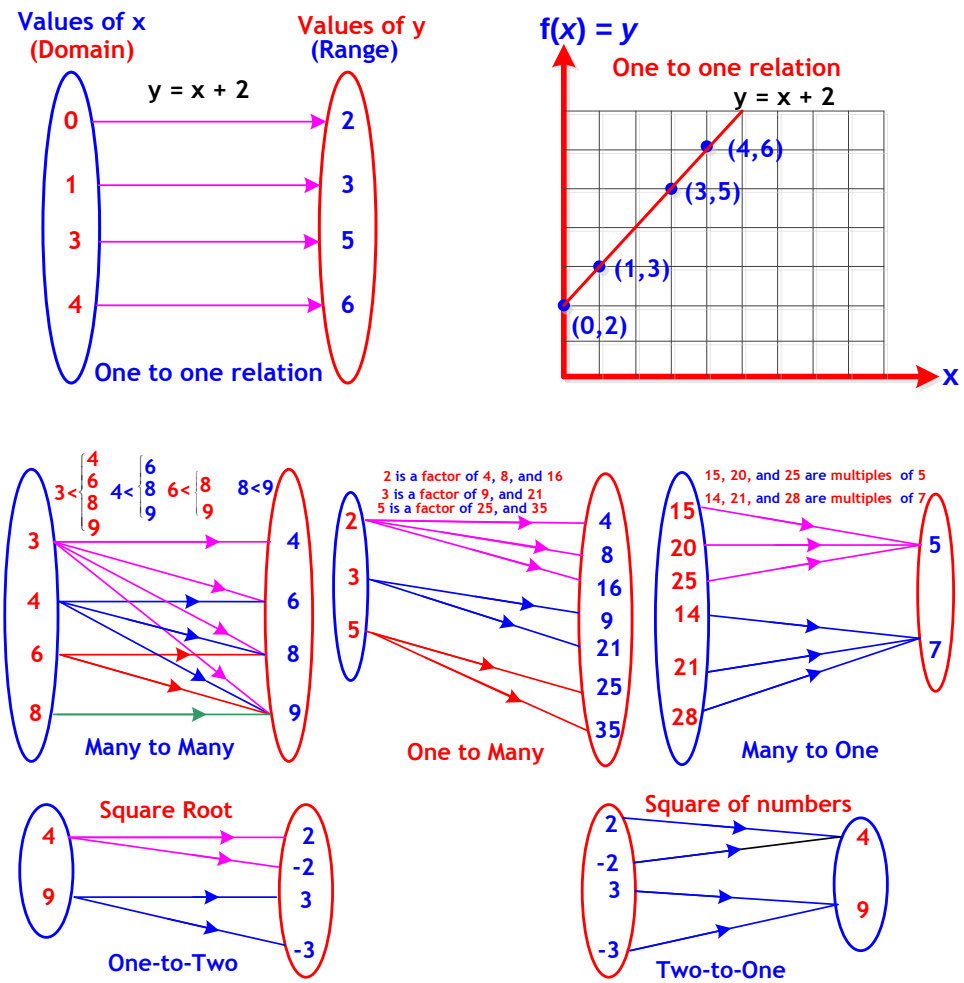
Chapter-35 Various Functions

A function is a relationship between the **inputs** and **outputs**. The values of **inputs** are the **domain**, and the values of the **outputs** are the **ranges**.

Note:

- The inverse function $f^{-1}(x)$: $\Rightarrow f^{-1}[f(x)]=x$ **and** $f\left[f^{-1}(x)\right]=x$
- If $f(x)=f^{-1}(x)$, then the function $f(x)$ is **self-inverse**.
- The range of $f(x)$ is the domain of $f^{-1}(x)$ and vice versa.
- The graphs of $f(x)$ **and** $f^{-1}(x)$ are **reflections** of each other with respect to the line $y=x$. This is because the roles of ***x*** and ***y*** are interchanged.
- **Only** one-to-one functions have **inverses**!

1. Arrow Diagram:



2. Function $y = f(x)$

Look at the following functions in great details. Try to understand, how the functions work.

Examples

$$f(x) = x + 3$$
$$\downarrow$$
$$f(y) = y + 3$$
$$\downarrow$$
$$f(t) = t + 3$$
$$\downarrow$$
$$f(x-2) = (x-2) + 3$$
$$\downarrow$$
$$f(a-b) = (a-b) + 3$$

Examples

$$f(x) = x^2 - 4x + 3$$
$$\downarrow$$
$$f(y) = y^2 - 4y + 3$$
$$\downarrow$$
$$f(t) = t^2 - 4t + 3$$
$$\downarrow$$
$$f(x-2) = (x-2)^2 - 4(x-2) + 3$$
$$\downarrow$$
$$f(a-b) = (a-b)^2 - 4(a-b) + 3$$

Examples

$$f(a-b) = (a-b)^2 - 4(a-b) + 3$$
$$\downarrow$$
$$f(x-2) = (x-2)^2 - 4(x-2) + 3$$
$$\downarrow$$
$$f(t) = t^2 - 4t + 3$$
$$\downarrow$$
$$f(y) = y^2 - 4y + 3$$
$$\downarrow$$
$$f(x) = x^2 - 4x + 3$$

Examples

$f(x) = x + 3$ Starting Point

$\downarrow$

$f(0) = 0 + 3 = 3$

$\downarrow$

$f(1) = 1 + 3 = 4$

$\downarrow$

$f(-2) = -2 + 3 = 1$

$\downarrow$

$f(-5) = -5 + 3 = -2$

Examples

$f(a-b) = (a-b)^2 - 4(a-b) + 3$ Starting Point

$\downarrow$

$f(-2) = (-2)^2 - 4(-2) + 3 = 4 + 8 + 3 = 15$

$\downarrow$

$f(0) = 0^2 - 4(0) + 3 = 0 - 0 + 3 = 3$

$\downarrow$

$f(3) = 3^2 - 4(3) + 3 = 9 - 12 + 3 = 0$

$\downarrow$

$f(-1) = (-1)^2 - 4(-1) + 3 = 1 + 4 + 3 = 8$

3. Important Information

Some books use the following notations for $f(x)$ as shown below.

(a) $f : x \mapsto \dfrac{x}{x-2}$ (b) $f : x \to \dfrac{x}{x-2}$ (c) $f(x) = \dfrac{x}{x-2}$

Note: In this note, $f(x)$ is used.

4. Composite Functions

Example: Consider the following functions:

$f(x) = 4x^2 - 1$ $\qquad$ $g(x) = 3x + 2$

Find:

(a) $fg(2)$ $\qquad$ (b) $fg(-2)$

(c) $fg(a)$ $\qquad$ (d) $fg(b)$

Ans:

(a) Find the **inside function first:** $g(2) = 3(2) + 2 = 8$, **then**

$fg(2) = f(g(2)) = f(8) = 4(8)^2 - 1 = 255$

(b) Find the **inside function first:** $g(-2) = 3(-2) + 2 = -4$, **then**

$fg(-2) = f(g(-2)) = f(-4) = 4(-4)^2 - 1 = 64 - 1 = 63$

(c) Find the **inside function first:** $g(a) = 3(a) + 2 = 3a + 2$, **then**

$fg(a) = f(g(a)) = f(3a+2) = 4(3a+2)^2 - 1 = 4(9a^2 + 12a + 4) - 1 = 36a^2 + 48a + 15$

(d) Find the **inside function first:** $g(b) = 3(b) + 2 = 3b + 2$, **then**

$fg(b) = f(g(b)) = f(3b+2) = 4(3b+2)^2 - 1 = 4(9b^2 + 12b + 4) - 1 = 36b^2 + 48b + 15$

Note: $f(g(x))$ **can be written as:** fg, **and** $g(f(x))$ **can be written as:** gf.

Example: Consider the following 2 different functions:

$f(x) = 4x^2 - 1$ $\qquad$ $g(x) = 3x + 2$

Find:

(a) $g(f(-2)) = gf(-2)$ $\qquad$ (b) $f(f(a)) = ff(a)$

(c) $g(g(b)) = gg(b)$ $\qquad$ (d) $f(g(b)) = fg(b)$

Ans:

(a) Find the **inside function first:** $f(-2) = 4(-2)^2 - 1 = 16 - 1 = 15$, **then**

$g(f(-2)) = g(15) = 3(15) + 2 = 45 + 2 = 47$

(b) Find the **inside function first:** $f(a) = 4(a)^2 - 1 = 4a^2 - 1$, **then**

$f(f(a)) = f(4a^2 - 1) = 4(4a^2 - 1)^2 - 1 = 4(16a^4 - 8a^2 + 1) - 1 = 64a^4 - 32a^2 + 3$

(c) Find the **inside function first:** $g(b) = 3(b) + 2 = 3b + 2$, **then**

$g(g(b)) = g(3a + 2) = 3(3b + 2) + 2 = 9b + 6 + 2 = 9b + 8$

(d) Find the **inside function first:** $g(b) = 3(b) + 2 = 3b + 2$, **then**

$f(g(b)) = f(3a+2) = 4(3b+2)^2 - 1 = 4(9b^2 + 12b + 4) = 36b^2 + 48b + 15$

Note: $f(f(x))$ **can be written as:** $ff = f^2$, **and** note that $ff = f^2 \neq [f(x)]^2$

Example: Consider the following functions:

$f(x)=2x+1$ $\quad g(x)=3x-7$ $\quad h(x)=2x+3$

Find:

(a) $f^2=ff$ (b) $\left[f(x)\right]^2$

(c) $f^3=fff$ (d) fgh

Ans:

(a) $f^2=ff=f\left[f(x)\right]=f\left(f(x)\right)=f(2x+1)=2(2x+1)+1=4x+2+1=4x+3$

(b) $\left[f(x)\right]^2=\left[2x+1\right]^2=4x^2+4x+1$

And $f^2=ff=f\left[f(x)\right]=f(2x+1)=2(2x+1)+1=4x+2+1=4x+3$

Therefore, $\left[f(x)\right]^2\neq f^2=f\left(f(x)\right)=f\left[f(x)\right]$

(c) $f^3=fff=ff\left(f(x)\right)=f\left[f\{f(x)\}\right]=f\left[f(2x+1)\right]=f(4x+3)=2(4x+3)+1$

And $\left[f(x)\right]^3\neq f^3$

(d) $fgh=fg\left(h(x)\right)=f\left[g(2x+3)\right]=f(6x+2)=2(6x+2)+1=12x+5$

Example: Consider the following functions:

$f(x)=2x-1$ $\quad g(x)=3x^2$ $\quad h(x)=\dfrac{2}{x}$

Find:

(a) $g^2=gg$ (b) $h^2=hh$

(c) fg (d) hfg

Ans:

(a) $g^2=gg=g\left[g(x)\right]=g\left(g(x)\right)=g\left(3x^2\right)=3\left(3x^2\right)^2=27x^4$

(b) $h^2=h\left(h(x)\right)=h\left[h(x)\right]=h\left(\dfrac{2}{x}\right)=\dfrac{2}{\frac{2}{x}}=2\div\dfrac{2}{x}=\dfrac{2x}{2}=x$

(c) $fg=f\left(g(x)\right)=f\left(3x^2\right)=2\left(3x^2\right)-1=6x^2-1$

(d) $hfg=hf\left(g(x)\right)=h\left[f\left(3x^2\right)\right]=h\left(6x^2-1\right)=\dfrac{2}{6x^2-1}$

Example: Consider the following functions:

$f(x)=2x^2 \qquad g(x)=3x-1$

Find the value of ***x*** such that $fg(x)=gf(x)$

Ans: $fg=f(g(x))=f(3x-1)=2(3x-1)^2=2(9x^2-6x+1)=18x^2-12x+2$

$gf=g(f(x))=g(2x^2)=3(2x^2)-1=6x^2-1$

Since $fg(x)=gf(x) \Rightarrow 18x^2-12x+2=6x^2-1$

$12x^2-12x+3=0$

$4x^2-4x+1=(2x-1)(2x-1)=0 \Rightarrow x_1=x_2=\frac{1}{2}$

Example: Consider the following functions:

$f(x)=2x-1 \qquad g(x)=3x+k$

Find the value of ***k*** if $fg(x)=gf(x)$

Ans: $fg=f(g(x))=f(3x+k)=2(3x+k)-1=6x+2k-1$

$gf=g(f(x))=g(2x-1)=3(2x-1)+k=6x-3+k$

Since $fg(x)=gf(x)$

$\Rightarrow 6x+2k-1=6x+k-3$

$2k-1=k-3$

$\Rightarrow k=-2$

Example: Consider the following functions:

$f(x)=2x-3 \qquad g(x)=3x+2$

Find the value of ***k*** if $fg(k)=gf(2k)$

Ans: $fg(k)=f(g(k))=f(3k+2)=2(3k+2)-3=6k+4-3=6k+1$

$gf(2k)=g(f(2k))=g(4k-3)=3(4k-3)+2=12k-9+2=12k-7$

Since $fg(k)=gf(2k)$

$\Rightarrow 6k+1=12k-7$

$6k=8$

$k=\frac{8}{6}=\frac{3}{4}$

Example: Consider the function $f(x)=x-3$, find:

(a) $ff(x)$ (b) $fff(x)$ (c) $f^n(x)$

(d) The function $g(x)$ such that $g^n(x)=2^n x$

(e) The function $h(x)$ such that $h^n(x)=\frac{x}{a^n}$

Ans:

(a) $ff(x)=f(f(x))=f(x-3)=(x-3)-3=x-6$

(b) $fff(x)=ff(f(x))=ff(x-3)=f[(x-3)-3]=f[x-6]=(x-6)-3=x-9$

(c) $f(x)=x-3=x-1\times 3$

$ff(x)=x-6=x-2\times 3$

$fff(x)=x-9=x-3\times 3$

.
.
.

$f^n(x)=x-9=x-3n$

(d) $g^n(x)=2^n x$

$\Rightarrow g(x)=2x$

(e) $h^n(x)=\frac{x}{a^n}$

$\Rightarrow h(x)=\frac{x}{a}$

Example: Consider the following functions:

$f(x)=2x-3$ $\quad g(x)=x^2+2$ $\quad h(x)=ax+b$

(a) Show that $fg=2x^2+1$ and find the function gf.

(b) Find ***a*** and ***b*** such that $fgh=18x^2+12x+3$

(c) Find the values of x for which $fgh(x)=9$

Ans:

(a) $fg(x)=f(g(x))=f(x^2+2)=2(x^2+2)-3=2x^2+4-3=2x^2+1$

$gf(x)=g(f(x))=g(2x-3)=(2x-3)^2+2=4x^2-12x+9+2=4x^2-12x+11$

(b) $fgh(x)=fg(ax+b)=f\left[(ax+b)^2+2\right]=2\left[(ax+b)^2+2\right]-3=2a^2x^2+4abx+2b^2+1$

$fgh(x)=2a^2x^2+4abx+2b^2+1=18x^2+12x+3$

$2a^2x^2+4abx+2b^2+1=18x^2+12x+3$ **...(1)**

$\Rightarrow 2a^2=18\Rightarrow a^2=9\Rightarrow a=3$ **and** $4ab=12\Rightarrow b=\frac{12}{4a}=\frac{12}{12}=1$

(c) $fgh=18x^2+12x+3=9\Rightarrow 18x^2+12x-6=0\Rightarrow 3x^2+2x-1=0$

$\Rightarrow 3x^2+2x-1=(3x-1)(x+1)=0\Rightarrow x=\frac{1}{3}$ **and** $x=-1$

Example: Consider the following functions:

$f(x)=4x^2-1$ $\quad g(x)=3x+2$ $\quad h(x)=x-2$

Find:

(a) $g(h(f(2)))=ghf(2)$ (b) $h(f(g(a)))=hfg(a)$ (c) $gh(f(b))=ghf(b)$

Ans:

(a) Find the **inside** function **first:** $f(2)=4(2)^2-1=16-1=15$, **then** $h(f(2))=h(15)=15-2=13$, **then** $g(13)=3(15)+2=45+2=47$

(b) Find the **inside** function **first:** $g(a)=3a+2$, **then** $f(g(a))=f(3a+2)=(3a+2)^2+2=9a^2+12a+4+2=9a^2+12a+6$, **then** $h(9a^2+12a+6)=9a^2+12a+6-2=9a^2+12a+4$

(c) Find the **inside** function **first:** $f(b)=4b^2-1$, **then**

$h(f(b))=h(4b^2-1)=4b^2-1-2=4b^2-3$, **then**

$gh(f(b))=g(4b^2-1)=3(4b^2-1)+2=12b^2-3+2=12b^2-1$

Example: Consider the following functions:

$f(x)=2x+3$ $\quad g(x)=3x-2$ $\quad h(x)=x+2$

Find: (a) $fg(3)$ (b) $gf(-2)$ (c) $fgh(0)$

(d) $hfg(2)$ (e) $gfh(-1)$ (f) $fhg(0)$

Ans:

(a) $g(3)=3(3)-2=9-2=7$

$\Rightarrow fg(3)=f(7)=2(7)+3=14+3=17$

OR $fg(x)=f(3x-2)=2(3x-2)+3=6x-4+3=6x-1$

$\Rightarrow fg(3)=6(3)-1=18-1=17$

(b) $f(-2)=2(-2)+3=-4+3=-1$

$\Rightarrow gf(-2)=g(-1)=3(-1)-2=-3-2=-5$

OR $gf(x)=g(2x+3)=3(2x+3)-2=6x+9-2=6x+7$

$\Rightarrow gf(-2)=6(-2)+7=-12+7=-5$

(c) $h(0)=0+2=2$

$\Rightarrow fgh(0)=fg(2)=f(6-2)=f(4)=2(4)+3=11$

OR $fgh(x)=fg(x+2)=f[3(x+2)-2]=f(3x+4)=2(3x+4)+3=6x+11$

$\Rightarrow fgh(0)=6(0)+11=11$

(d) $g(2)=6-2=4$

$\Rightarrow hfg(2)=hf(4)=h(8+3)=h(11)=11+2=13$

OR $hfg(x)=hf(3x-2)=h\left[2(3x-2)+3\right]=h(6x-1)=6x-1+2=6x+1$

$\Rightarrow hfg(2)=6(2)+1=13$

(e) $h(-1)=-1+2=1$

$\Rightarrow gfh(-1)=gf(1)=g(2+3)=g(5)=3(5)-2=15-2=13$

OR $gfh(x)=gf(x+2)=g\left[2(x+2)+3\right]=g(2x+7)=3(2x+7)-2=6x+19$

$\Rightarrow gfh(-1)=-6+19=13$

(f) $g(0)=0-2=-2$

$\Rightarrow fhg(0)=fh(-2)=f(-2+2)=f(0)=0+3=3$

OR $fhg(x)=fh(3x-2)=f\left[3x-2+2\right]=f(3x)=2(3x)+3=6x+3$

$\Rightarrow fhg(0)=0+3=3$

Example: The functions ***f(x)***, ***g(x)***, and ***h(x)*** are defined as follows:

$f(x)=\sqrt{x-2} \qquad x\geq 2$

$g(x)=2x-3 \qquad x\in\Re$

$h(x)=x^2 \qquad x\geq 0$

Find (a) $fg(x)$ (b) $hg(x)$ (c) $hgf(x)$

Ans: **(a)** $f(x)=\sqrt{x-2} \quad x\geq 2$ **and** $g(x)=2x-3 \quad x\in\Re$

$fg(x)=f\left[g(x)\right]=f(2x-3)=\sqrt{2x-3-2}=\sqrt{2x-5}$

(b) $g(x)=2x-3 \quad x\in\Re$ **and** $h(x)=x^2 \quad x\geq 0$

$hg(x)=h\left[g(x)\right]=h(2x-3)=(2x-3)^2$

(c) $f(x)=\sqrt{x-2} \quad x\geq 2$, $g(x)=2x-3 \quad x\in\Re$, **and** $h(x)=x^2 \quad x\geq 0$

$hgf(x)=hg\left[f(x)\right]=hg\left(\sqrt{x-2}\right)=h\left(2\sqrt{x-2}-3\right)=\left(2\sqrt{x-2}-3\right)^2$

$hgf(x)=4(x-2)-12\sqrt{x-2}+9=4x-8-12\sqrt{x-2}+9=4x+1-12\sqrt{x-2}$

Example: The functions ***p(x)***, ***q(x)***, and ***r(x)*** are defined as follows:

$$p(x) = ax \qquad q(x) = x + b \qquad r(x) = \frac{1}{x}$$

Find (a) pqr (b) qpr (c) rpq (d) prq

(e) p^2q^2 (f) q^2p^2 (g) $(pq)^2$ (h) $(qp)^2$

Ans: **(a)** $pqr = pq\left(\frac{1}{x}\right) = p\left(\frac{1}{x} + b\right) = p\left(\frac{1+bx}{x}\right) = a\left(\frac{1+bx}{x}\right)$

(b) $qpr = qp\left(\frac{1}{x}\right) = q\left(\frac{a}{x}\right) = \frac{a}{x} + b$

(c) $rpq = rp(x+b) = r(ax+ab) = \frac{1}{a(x+b)}$

(d) $prq = pr(x+b) = p\left(\frac{1}{x+b}\right) = \frac{a}{x+b}$

(e) $p^2q^2 = ppqq(x) = ppq(ax) = pp(x+b+b) = pp(x+2b)$

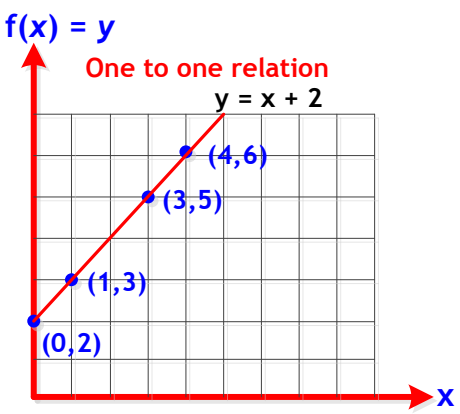
$= p[a(x+2b)] = p(ax+2ab) = a(ax+2ab) = a^2x + 2a^2b$

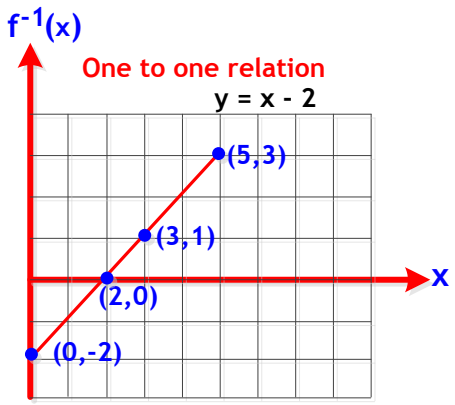
(f) $q^2p^2 = qqpp(x) = qqp(ax) = qq(a^2x) = q(a^2x+b) = a^2x+b+b = a^2x+2b$

(g) $(pq)^2 = [pq(x)]^2 = [p(x+b)]^2 = (ax+ab)^2$

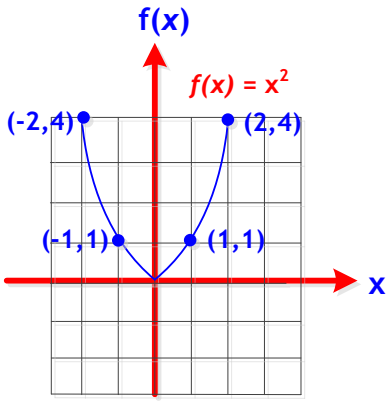
(h) $(qp)^2 = [qp(x)]^2 = [q(ax)]^2 = (ax+b)^2$

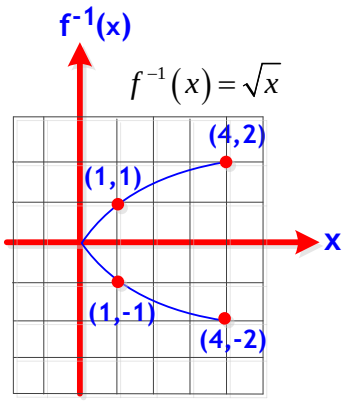
Example: Consider the following graphs of $f(x)$ **and** $f^{-1}(x)$.

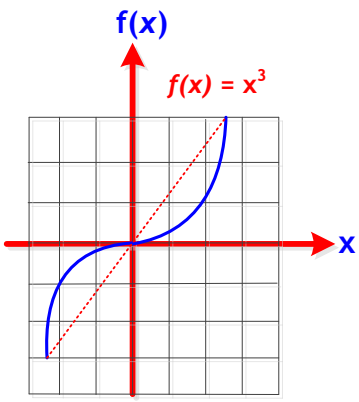


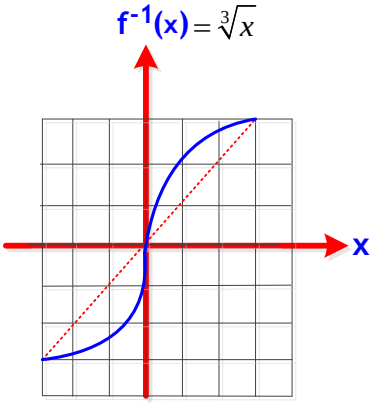


Note: Values of x and y are interchanged.









Note: Values of x and y are interchanged.

5. Inverse Functions

Example: Work out the **inverses** of the following functions:

(a) $f(x)=4x$ (b) $f(x)=3x+2$ (c) $f(x)=7-x^3$

Ans: **(a)**

Initial Function: $f(x)=4x$

Step-1: **Replace f(x) with y:** $y=4x$

Step-2: **Switch x and y:** $x=4y$

Step-3: **Solve for y:** $y=\frac{x}{4}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\frac{x}{4}$

Note: $f^{-1}\left[f(x)\right]=f^{-1}\left[4x\right]=\frac{4x}{4}=x$ **OK**

OR $f\left[f^{-1}(x)\right]=f\left[\frac{x}{4}\right]=4\left(\frac{x}{4}\right)=x$ **OK**

(b)

Initial Function: $f(x)=3x+2$

Step-1: **Replace f(x) with y:** $y=3x+2$

Step-2: **Switch x and y:** $x=3y+2$

Step-3: **Solve for y:** $y=\dfrac{x-2}{3}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\dfrac{x-2}{3}$

(c)

Initial Function: $f(x)=7-x^3$

Step-1: **Replace f(x) with y:** $y=7-x^3$

Step-2: **Switch x and y:** $x=7-y^3$

Step-3: **Solve for y:** $y=\sqrt[3]{7-x}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\sqrt[3]{7-x}$

Note: $f^{-1}\left[f(x)\right]=f^{-1}\left(7-x^3\right)=\sqrt[3]{7-7+x^3}=\sqrt[3]{x^3}=x$ **OK**

OR $f\left[f^{-1}(x)\right]=f\left(\sqrt[3]{7-x}\right)=7-\left(\sqrt[3]{7-x}\right)^3=7-7+x=x$ **OK**

Example: The functions ***f(x)***, ***g(x)***, and ***h(x)*** are defined as follows:

$f(x)=\sqrt{x-2}$ $\quad x\geq 2$ $\quad g(x)=2x-3$ $\quad x\in\Re$ $\quad h(x)=x^2$ $\quad x\geq 0$

Find (a) $g^{-1}(x)$ (b) $f^{-1}(x)$ (c) $h^{-1}(x)$

Ans: **(a)**

Initial Function: $g(x) = 2x - 3$

Step-1: **Replace g(x) with y:** $y = 2x - 3$

Step-2: **Switch x and y:** $x = 2y - 3$

Step-3: **Solve for y:** $y = \dfrac{x+3}{2}$

Step-4: **Replace y with $g^{-1}(x)$:** $g^{-1}(x) = \dfrac{x+3}{2}$

(b)

Initial Function: $f(x) = \sqrt{x-2}$

Step-1: **Replace f(x) with y:** $y = \sqrt{x-2}$

Step-2: **Switch x and y:** $x = \sqrt{y-2}$

Step-3: **Solve for y:** $y = x^2 + 2$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x) = x^2 + 2$

(c)

Initial Function: $h(x) = x^2$

Step-1: **Replace *h(x)* with y:** $y = x^2$

Step-2: **Switch x and y:** $x = y^2$

Step-3: **Solve for y:** $y = \sqrt{x}$

Step-4: **Replace y with $h^{-1}(x)$:** $h^{-1}(x) = \sqrt{x}$

Example: Which one of the functions below **does not have** an inverse function?

(a) $f(x) = \dfrac{1}{x+1}$ (b) $f(x) = x^3 - 2$

(c) $f(x) = x^2 - 3 \quad x \geq 0$ (d) $f(x) = x^3 + 3x$

Ans: (a)

Initial Function: $f(x)=\dfrac{1}{x+1}$

Step-1: Replace f(x) with y: $y=\dfrac{1}{x+1}$

Step-2: Switch x and y: $x=\dfrac{1}{y+1}$

Step-3: Solve for y: $y=\dfrac{1-x}{x}$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=\dfrac{1-x}{x}$

Note: $f^{-1}\left[f(x)\right]=f^{-1}\left[\dfrac{1}{x+1}\right]=\dfrac{1-\dfrac{1}{x+1}}{\dfrac{1}{x+1}}=\dfrac{\dfrac{x+1-1}{x+1}}{\dfrac{1}{x+1}}=\dfrac{\dfrac{x}{x-1}}{\dfrac{1}{x-1}}=x$ **OK**

OR $f\left[f^{-1}(x)\right]=f\left[\dfrac{1-x}{x}\right]=\dfrac{1}{\dfrac{1-x}{x}+1}=\dfrac{1}{\dfrac{1-x+x}{x}}=\dfrac{1}{\dfrac{1}{x}}=x$ **OK**

(b)

Initial Function: $f(x)=x^3-2$

Step-1: Replace f(x) with y: $y=x^3-2$

Step-2: Switch x and y: $x=y^3-2$

Step-3: Solve for y: $y=\sqrt[3]{x+2}$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=\sqrt[3]{x+2}$

Note: $f^{-1}\left[f(x)\right]=f^{-1}\left[x^3-2\right]=\sqrt[3]{x^3-2-2}=\sqrt[3]{x^3}=x$ **OK**

OR $f\left[f^{-1}(x)\right]=f\left[\sqrt[3]{x+2}\right]=\left(\sqrt[3]{x+2}\right)^3-2=x+2-2=x$ **OK**

(c)

Initial Function: $f(x) = x^2 - 3$

Step-1: **Replace f(x) with y:** $y = x^2 - 3$

Step-2: **Switch x and y:** $x = y^2 - 3$

Step-3: **Solve for y:** $y = \sqrt{x+3}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x) = \sqrt{x+3}$

Note: $f^{-1}[f(x)] = f^{-1}[\sqrt{x+3}] = (\sqrt{x+3})^2 - 3 = x + 3 - 3 = x$ **OK**

OR $f[f^{-1}(x)] = f[\sqrt{x+3}] = (\sqrt{x+3})^2 - 3 = x + 3 - 3 = x$ **OK**

(d)

Initial Function: $f(x) = x^3 + 3x$

Step-1: **Replace f(x) with y:** $y = x^3 + 3x$

Step-2: **Switch x and y:** $x = y^3 + 3y$

Step-3: **Solve for y:** **Not Possible**

Step-4: **Therefore, $f^{-1}(x)$:** **Not Possible**

Example: Work out the inverses of the following functions:

(a) $f(x) = 6x$ (b) $f(x) = 3 + 4x^3$ (c) $f(x) = 1 - 3x$

Ans: **(a)**

Initial Function: $f(x) = 6x$

Step-1: **Replace f(x) with y:** $y = 6x$

Step-2: **Switch x and y:** $x = 6y$

Step-3: **Solve for y:** $y = \frac{x}{6}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x) = \frac{x}{6}$

(b)

Initial Function: $f(x)=3+4x^3$

Step-1: **Replace f(x) with y:** $y=3+4x^3$

Step-2: **Switch x and y:** $x=3+4y^3$

Step-3: **Solve for y:** $y=\sqrt[3]{\frac{x-3}{4}}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\sqrt[3]{\frac{x-3}{4}}$

Note: $f^{-1}\left[f(x)\right]=f^{-1}\left(3+4x^3\right)=\sqrt[3]{\frac{3+4x^3-3}{4}}=\sqrt[3]{x^3}=x$ **OK**

OR $f\left[f^{-1}(x)\right]=f\left(\sqrt[3]{\frac{x-3}{4}}\right)=3+\left(\sqrt[3]{\frac{x-3}{4}}\right)^3=3+x-3=x$ **OK**

(c)

Initial Function: $f(x)=1-3x$

Step-1: **Replace f(x) with y:** $y=1-3x$

Step-2: **Switch x and y:** $x=1-3y$

Step-3: **Solve for y:** $y=\frac{1-x}{3}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\frac{1-x}{3}$

Note: $f^{-1}\left[f(x)\right]=f^{-1}(1-3x)=\frac{1-1+3x}{3}=\frac{3x}{3}=x$ **OK**

OR $f\left[f^{-1}(x)\right]=f\left(\frac{1-x}{3}\right)=1-3\left(\frac{1-x}{3}\right)=1-1+x=x$ **OK**

Example: Work out the inverses of the following functions:

(a) $f(x)=\frac{x+1}{2}$ (b) $f(x)=\frac{x-4}{3-x}$

(c) $f(x)=\frac{4x+5}{x+2}$

Ans: (a)

Initial Function: $f(x)=\frac{x+1}{2}$

Step-1: Replace f(x) with y: $y=\frac{x+1}{2}$

Step-2: Switch x and y: $x=\frac{y+1}{2}$

Step-3: Solve for y: $y=2x-1$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=2x-1$

Note: The following steps are just for testing. They are not required.

Verify: $f^{-1}[f(x)]=f^{-1}\left(\frac{x+1}{2}\right)=2\left(\frac{x+1}{2}\right)-1=\frac{2x+2-2}{2}=\frac{2x}{2}=x$ OK

OR $f[f^{-1}(x)]=f(2x+1)=\frac{2x-1+1}{2}=\frac{2x}{2}=x$ OK

(b)

Initial Function: $f(x)=\frac{x-4}{3-x}$

Step-1: Replace f(x) with y: $y=\frac{x-4}{3-x}$

Step-2: Switch x and y: $x=\frac{y-4}{3-y}$

Step-3: Solve for y: $y=\frac{3x+4}{x+1}$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=\frac{3x+4}{x+1}$

Note: The following steps are just for testing. They are not required.

Verify: $f^{-1}[f(x)]=f^{-1}\left(\frac{x-4}{3-x}\right)=\frac{3\left(\frac{x-4}{3-x}\right)+4}{\frac{x-4}{3-x}+1}=\frac{\frac{3x-12+12-4x}{3-x}}{\frac{x-4+3-x}{3-x}}=\frac{-x}{-1}=x$ OK

OR $f[f^{-1}(x)]=f\left(\frac{3x+4}{x+1}\right)=\frac{\left(\frac{3x+4}{x+1}\right)-4}{3-\frac{3x+4}{x+1}}=\frac{\frac{3x+4-4x-4}{x+1}}{\frac{3x+3-3x-4}{x+1}}=\frac{-x}{-1}=x$ OK

(c)

Initial Function: $f(x)=\dfrac{4x+5}{x+2}$

Step-1: Replace f(x) with y: $y=\dfrac{4x+5}{x+2}$

Step-2: Switch x and y: $x=\dfrac{4y+5}{y+2}$

Step-3: Solve for y: $y=\dfrac{5-2x}{x-4}$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=\dfrac{5-2x}{x-4}$

Note: The following steps are just for testing. They are not required.

Verify: $f^{-1}\left[f(x)\right]=f^{-1}\left(\dfrac{4x+5}{x+2}\right)=\dfrac{5-2\left(\dfrac{4x+5}{x+2}\right)}{\dfrac{4x+5}{x+2}-4}=\dfrac{\dfrac{5x+10-8x-10}{x+2}}{\dfrac{4x+5-4x-8}{x+2}}=\dfrac{-3x}{-3}=x$ OK

OR $f\left[f^{-1}(x)\right]=f\left(\dfrac{5-2x}{x-4}\right)=\dfrac{4\left(\dfrac{5-2x}{x-4}\right)+5}{\dfrac{5-2x}{x-4}+2}=\dfrac{\dfrac{20-8x+5x-20}{x-4}}{\dfrac{5-2x+2x-8}{x-4}}=\dfrac{-3x}{-3}=x$ OK

Example: If $f(x)=\dfrac{2x-3}{x+1}$, find $f^{-1}(3)$.

Ans:

Initial Function: $f(x)=\dfrac{2x-3}{x+1}$

Step-1: Replace f(x) with y: $y=\dfrac{2x-3}{x+1}$

Step-2: Switch x and y: $x=\dfrac{2y-3}{y+1}$

Step-3: Solve for y: $y=\dfrac{x+3}{2-x}$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=\dfrac{x+3}{2-x}$

Therefore, $f^{-1}(3)=\dfrac{3+3}{2-3}=-6$

Example: By **finding** the **inverse** of $f(x)=\dfrac{2x-3}{x+1}$, **solve** the equation $\dfrac{2x-3}{x+1}=4$.

Ans:

Initial Function: $f(x)=\dfrac{2x-3}{x+1}$

Step-1: **Replace f(x) with y:** $y=\dfrac{2x-3}{x+1}$

Step-2: **Switch x and y:** $x=\dfrac{2y-3}{y+1}$

Step-3: **Solve for y:** $y=\dfrac{x+3}{2-x}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\dfrac{x+3}{2-x}$

Since $f^{-1}(4)=\dfrac{4+3}{2-4}=-\dfrac{7}{2}$, **Then, the solution of** $\dfrac{2x-3}{x+1}=4$ **is** $x=-\dfrac{7}{2}$

Example: If $f(x)=3x-5$, **and** $g(x)=2x+1$, **then find:**

(a) $f^{-1}(x)$. (b) $g^{-1}(x)$.

(c) $(fg)^{-1}$. (d) $(gf)^{-1}$.

(e) $f^{-1}g^{-1}$. (f) $g^{-1}f^{-1}$.

Ans: (a)

Initial Function: $f(x)=3x-5$

Step-1: **Replace f(x) with y:** $y=3x-5$

Step-2: **Switch x and y:** $x=3y-5$

Step-3: **Solve for y:** $y=\dfrac{x+5}{3}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\dfrac{x+5}{3}$

(b)

Initial Function: $g(x)=2x+1$

Step-1: Replace g(x) with y: $y=2x+1$

Step-2: Switch x and y: $x=2y+1$

Step-3: Solve for y: $y=\dfrac{x-1}{2}$

Step-4: Replace y with $g^{-1}(x)$: $g^{-1}(x)=\dfrac{x-1}{2}$

(c)

$f(x)=3x-5 \qquad g(x)=2x+1$

$fg=f\left[g(x)\right]=f(2x+1)=3(2x+1)-5=6x-2$

Let $fg=h(x)=6x-2$

Initial Function: $h(x)=6x-2$

Step-1: Replace *h(x)* with y: $y=6x-2$

Step-2: Switch x and y: $x=6y-2$

Step-3: Solve for y: $y=\dfrac{x+2}{6}$

Step-4: Replace y with $h^{-1}(x)$: $h^{-1}(x)=\dfrac{x+2}{6}$

Therefore, $(fg)^{-1}=h^{-1}(x)=\dfrac{x+2}{6}$

(d)

$f(x)=3x-5 \qquad g(x)=2x+1$

$gf=g\left[f(x)\right]=g(3x-5)=2(3x-5)+1=6x-9$

Let $gf=h(x)=6x-9$

Initial Function: $h(x) = 6x - 9$

Step-1: **Replace *h(x)* with y:** $y = 6x - 9$

Step-2: **Switch x and y:** $x = 6y - 9$

Step-3: **Solve for y:** $y = \frac{x+9}{6}$

Step-4: **Replace y with $h^{-1}(x)$:** $h^{-1}(x) = \frac{x+9}{6}$

Therefore, $(gf)^{-1} = h^{-1}(x) = \frac{x+9}{6}$

(e)

$f^{-1}(x) = \frac{x+5}{3}$ **and** $g^{-1}(x) = \frac{x-1}{2}$ from answer **(a)** and **(b)**

$$f^{-1}g^{-1} = f^{-1}\left[g^{-1}(x)\right] = f^{-1}\left(\frac{x-1}{2}\right) = \frac{\frac{x-1}{2}+5}{3} = \frac{\frac{x-1+10}{2}}{3} = \frac{x+9}{6}$$

$$\Rightarrow f^{-1}g^{-1} = \frac{x+9}{6}$$

And $(gf)^{-1} = \frac{6x-9}{6}$ ... from answer **(d)**

Therefore, $(gf)^{-1} = f^{-1}g^{-1} = \frac{x+9}{6}$

(f)

$f^{-1}(x) = \frac{x+5}{3}$ **and** $g^{-1}(x) = \frac{x-1}{2}$ from answer **(a)** and **(b)**

$$g^{-1}f^{-1} = g^{-1}\left[f^{-1}(x)\right] = g^{-1}\left(\frac{x+5}{3}\right) = \frac{\frac{x+5}{3}-1}{2} = \frac{\frac{x+5-3}{3}}{2} = \frac{x+2}{6}$$

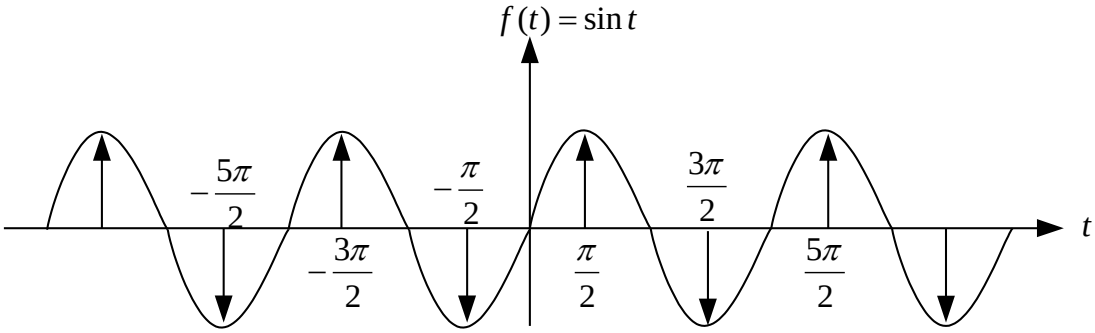
$$\Rightarrow g^{-1}f^{-1} = \frac{x+2}{6}$$

And $(fg)^{-1} = \frac{x+2}{6}$ ... from answer **(a)**

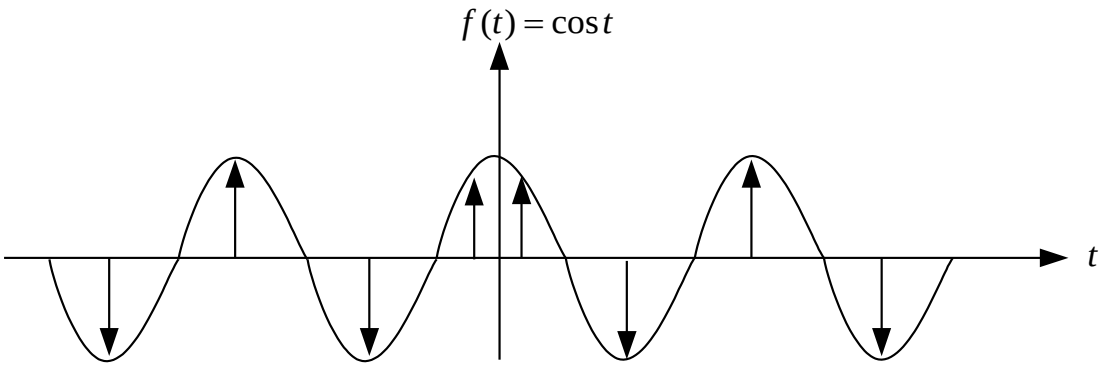
Therefore, $(fg)^{-1} = g^{-1}f^{-1} = \frac{x+2}{6}$

6. Odd and Even Functions

(1) **Odd function:** A function is said to be odd, when it satisfies the condition that $f(-t) = -f(t)$. For example, **sine** is an odd function as it can be seen from the following diagram.



(2) **Even function**: A function is said to be even, when it satisfies the condition that $f(-t) = f(t)$. For example, cosine is an even function as it can be seen from the following diagram.



Example: Which one of the following functions is odd?

(a) $f(x) = 3 - x$ (b) $f(x) = \cos x$

(c) $f(x) = 16x - x^5$ (d) $f(x) = x^3 + 5x^2$

Ans: **(a)** $f(x) = 3 - x \Rightarrow f(-x) = 3 + x \Rightarrow f(-x) \neq -f(x)$ **Not Odd**

(b) $f(x) = \cos x \Rightarrow f(-x) = \cos(-x) = \cos x \Rightarrow f(-x) \neq -f(x)$ **Not Odd**

(c) $f(x) = 16x - x^5 \Rightarrow f(-x) = 16(-x) - (-x)^5 = -16x + x^5 \Rightarrow f(-x) = -f(x)$ **Odd**

(d) $f(x) = x^3 + 5x^2 \Rightarrow f(-x) = (-x)^3 + 5(-x)^2 = -x^3 + 5x^2 \Rightarrow f(-x) \neq -f(x)$ **Not Odd**

Example: Which one of the following functions is **not** even?

(a) $f(x)=5x^4$ (b) $f(x)=(x+2)^2$

(c) $f(x)=\dfrac{1}{4+x^2}$ (d) $f(x)=x^6-3x^2+1$

Ans: **(a)** $f(x)=5x^4 \Rightarrow f(-x)=5(-x)^4=5x^4 \Rightarrow f(-x)=f(x)$ **Even**

(b) $f(x)=(x+2)^2 \Rightarrow f(-x)=(-x+2)^2 \Rightarrow f(-x)\neq -f(x)$ **Not Even**

(c) $f(x)=f(x)=\dfrac{1}{4+x^2} \Rightarrow f(-x)=\dfrac{1}{4+(-x)^2}=\dfrac{1}{4+x^2} \Rightarrow f(-x)=f(x)$ **Even**

(d) $f(x)=x^6-3x^2+1 \Rightarrow f(-x)=(-x)^6-3(-x)^2+1=x^6-3x^2+1 \Rightarrow f(-x)=f(x)$ **Even**

Example: Show that $f(x)=\dfrac{x}{x-1}$, where $x\neq 1$ is self-inverse.

Ans:

Initial Function: $f(x)=\dfrac{x}{x-1}$

Step-1: **Replace f(x) with y:** $y=\dfrac{x}{x-1}$

Step-2: **Switch x and y:** $x=\dfrac{y}{y-1}$

Step-3: **Solve for y:** $y=\dfrac{x}{x-1}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\dfrac{x}{x-1}$

Example: If $f(x-2)=x^2-4x+5$, then **find** $f(x)$.

Ans: $f(x-2)=x^2-4x+5=(x-2)^2-4+5=(x-2)^2+1$

Since, $f(x-2)=(x-2)^2+1 \Rightarrow f(x)=x^2+1$

Example: If $f(x+2)=x^2+4x+5$, then **find** $f(x)$.

Ans: $f(x+2)=x^2+4x+5=(x+2)^2-4+5=(x+2)^2+1$

Since, $f(x+2)=(x+2)^2+1 \Rightarrow f(x)=x^2+1$

EXERCISES

1. The function f is defined by:

$f: x \mapsto \frac{x}{x-2}$, **or** $f(x)=\frac{x}{x-2}$ **and** $g: x \mapsto \frac{x}{3}$, **or** $g(x)=\frac{x}{3}$.

(a) Find an expression for $f^{-1}(x)$

(b) Calculate $gf(1.5)$

(c) Find the value of x for which $g(x)=f(x)+\frac{1}{2}$.

2. Work out $f^{-1}(x)$ for each of the following functions:

(a) $f(x)=\frac{3}{4x-4}$ (b) $f(x)=\frac{x+1}{x+2}$ (c) $f(x)=5-2x^2$

3. State whether the following functions are **odd**, **even** or **neither**.

(a) $f(x)=(x-1)^2$ (b) $f(x)=\frac{1}{x}$ (c) $f(x)=1+\frac{1}{x}$

(d) $f(x)=\sin x$ (e) $f(x)=\cos x$ (f) $f(x)=x^3+3$

4. Given that $f(x)=\frac{3x+b}{x-3}$, find $f^{-1}(x)$.

5. Given that $f(x)=\log_a x$, find $f^{-1}(x)$.

6. Given that $f(x)=4x^2+5$, find $g(x)=f^{-1}(x)$.

7. Given that $f(x)=4x+5$, **and** $g(x)=5x-k$. Find the value of ***k*** for which $fg=gf$.

8. Given that $f(x)=\frac{1}{x+2}$, **find** $ff(3)$ **and** $f^{-1}(3)$.

9. If $f(x)=\sqrt{x}$, **and** $g(x)=x+5$, then express the following functions in terms of ***f*(x)** and ***g*(x).**

(a) $f_1(x)=\sqrt{x+5}$ (b) $f_2(x)=x-5$ (c) $f_3(x)=x+10$

(d) $f_4(x)=\sqrt{x}+10$ (e) $f_5(x)=x^2+5$

10. If $f(k)=9^{2k}-5^{2k}$, **where** k is a **positive** integer

(a) find $f(0)$ and $f(1)$

(b) find $f(k+1)$

(c) Prove that $f(k+1)-25f(k)=56(9^{2k})$

(d) Hence, using induction, prove that $f(k)$ is always divisible by 7.

11. If $f(x)=3x-7$ **and** $g(x)=x^2$, **find:**

(a) $fg(x)$ (b) $gf(x)$ (c) $f^{-1}(-4)$

(d) $g^{-1}(x)$ (e) $f^{-1}g^{-1}(x)$

12. If $f(x)=7x+3$ **and** $g(x)=x^3$, **find:**

(a) $fg(x)$ (b) $gf(x)$ (c) $f^{-1}(x)$

(d) $g^{-1}(x)$ (e) $f^{-1}g^{-1}(x)$ (f) $(fg)^{-1}(-6)$

13. Consider the following functions:

$f(x)=2x$ $g(x)=x^2$ $h(x)=x-1$

Find: (a) fgh (b) fhg (c) gfh

(d) ghf (e) hfg (f) hgf

14. Consider the following functions:

$f(x)=2x$ $g(x)=x^2$ $h(x)=x-1$

Find: (a) $fgh(0)$ (b) $fhg(-2)$ (c) $gfh(5)$

(d) $ghf(a)$ (e) $hfg(b)$ (f) $hgf(k)$

15. Consider the following functions, **find** $f(x)$.

(a) $f(x-1)=x^2+x+1$ (b) $f(x+2)=x^2+x+-6$

(c) $f(x-3)=x^2-7x+10$ (d) $f\left(\frac{x}{2}\right)=\frac{1}{4}x^2+\frac{3}{2}x+-4$

EXERCISES and SOLUTIONS

1. The function f is defined by:

$f: x \mapsto \frac{x}{x-2}$, **or** $f(x)=\frac{x}{x-2}$ **and** $g: x \mapsto \frac{x}{3}$, **or** $g(x)=\frac{x}{3}$.

(a) Find an expression for $f^{-1}(x)$

(b) Calculate $gf(1.5)$

(c) Find the value of x for which $g(x)=f(x)+\frac{1}{2}$.

Ans:

(a)

Initial Function: $f(x)=\frac{x}{x-2}$

Step-1: **Replace f(x) with y:** $y=\frac{x}{x-2}$

Step-2: **Switch x and y:** $x=\frac{y}{y-2}$

Step-3: **Solve for y:** $y=\frac{x}{x-2}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\frac{x}{x-2}$

(b)

$$gf(1.5)=g\left(\frac{1.5}{1.5-2}\right)=g\left(\frac{1.5}{-0.5}\right)=g(-3)=\frac{-3}{3}=-1$$

OR $$gf=gf(x)=g\left(\frac{x}{x-2}\right)=\frac{\frac{x}{x-2}}{3}=\frac{x}{3x-6}$$

$$\Rightarrow gf(1.5)=\left.\frac{x}{3x-6}\right|_{x=1.5}=\frac{1.5}{3(1.5)-6}=\frac{1.5}{4.5-6}=\frac{1.5}{-1.5}=-1$$

(c)

$$g(x)=f(x)-2$$

$$\frac{x}{3}=\frac{x}{x-2}-2=\frac{x-2(x-2)}{x-2}=\frac{-x+4}{x-2}\Rightarrow 3(-x+4)=x(x-2)$$

$$\Rightarrow x^2+x-12=(x-3)(x+4)=0\Rightarrow x=3 \text{ and } -4$$

2. Work out $f^{-1}(x)$ for each of the following functions:

(a) $f(x)=\dfrac{3}{4x-4}$ (b) $f(x)=\dfrac{x+1}{x+2}$ (c) $f(x)=5-2x^2$

Ans:

(a)

Initial Function: $f(x)=\dfrac{3}{4x-4}$

Step-1: Replace f(x) with y: $y=\dfrac{3}{4x-4}$

Step-2: Switch x and y: $x=\dfrac{3}{4y-4}$

Step-3: Solve for y: $y=\dfrac{4x+3}{4x}$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=\dfrac{4x+3}{4x}$

(b)

Initial Function: $f(x)=\dfrac{x+1}{x+2}$

Step-1: Replace f(x) with y: $y=\dfrac{x+1}{x+2}$

Step-2: Switch x and y: $x=\dfrac{y+1}{y+2}$

Step-3: Solve for y: $y=\dfrac{1-2x}{x-1}$

Step-4: Replace y with $f^{-1}(x)$: $f^{-1}(x)=\dfrac{1-2x}{x-1}$

(c)

Initial Function: $f(x)=5-2x^2$

Step-1: **Replace f(x) with y:** $y=5-2x^2$

Step-2: **Switch x and y:** $x=5-2y^2$

Step-3: **Solve for y:** $y=\sqrt{\frac{5-x}{2}}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\sqrt{\frac{5-x}{2}}$

3. State whether the following functions are **odd**, **even** or **neither**.

(a) $f(x)=(x-1)^2$ (b) $f(x)=\frac{1}{x}$ (c) $f(x)=1+\frac{1}{x}$

(d) $f(x)=\sin x$ (e) $f(x)=\cos x$ (f) $f(x)=x^3+3$

Ans:

For **odd function:** ***f*(-*x*) = -*f*(*x*)** and for **even function**: ***f*(-*x*) = *f*(*x*)**

(a) $f(x)=(x-1)^2$

$f(-x)=(-x-1)^2=x^2+2x+1$

$-f(x)=-(x-1)^2=-(x^2-2x+1)=-x^2+2x-1$

$f(-x)\neq -f(x)\Rightarrow$ **Not odd**

And also $f(-x)\neq f(x)\Rightarrow$ **Not Even, therefore neither.**

(b) $f(x)=\frac{1}{x}\Rightarrow f(-x)=\frac{1}{-x}=-\frac{1}{x}=-\left(\frac{1}{x}\right)\Rightarrow f(-x)=-f(x)$ **Odd**

(c) $f(x)=1+\frac{1}{x}\Rightarrow f(-x)=1+\frac{1}{-x}=1-\frac{1}{x}=-\left(-1+\frac{1}{x}\right)\Rightarrow f(-x)\neq -f(x)$

And also $f(-x)\neq f(x)\Rightarrow$ **Not Even, therefore neither.**

(d) $f(x)=\sin x\Rightarrow f(-x)=\sin(-x)=-\sin x\Rightarrow f(-x)=-f(x)$ **Odd**

(e) $f(x)=\cos x\Rightarrow f(-x)=\cos(-x)=\cos x\Rightarrow f(-x)=f(x)$ **Even**

(f) $f(x)=x^3+3\Rightarrow f(-x)=(-x)^3+3=-x^3+3\Rightarrow f(-x)\neq -f(x)\ and\ f(-x)\neq f(x)$ a **neither**

4. Given that $f(x)=\dfrac{3x+b}{x-3}$, **find** $f^{-1}(x)$.

Initial Function: $f(x)=\dfrac{3x+b}{x-3}$

Step-1: **Replace f(x) with y:** $y=\dfrac{3x+b}{x-3}$

Step-2: **Switch x and y:** $x=\dfrac{3y+b}{y-3}$

Step-3: **Solve for y:** $y=\dfrac{3x+b}{x-3}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\dfrac{3x+b}{x-3}$

Note: $$f^{-1}\left[f(x)\right]=f^{-1}\left[\frac{3x+b}{x-3}\right]=\frac{3\left(\frac{3x+b}{x-3}\right)+b}{\frac{3x+b}{x-3}-3}=\frac{\frac{9x+3b+bx-3b}{x-3}}{\frac{3x+b-3x+9}{x-3}}=\frac{\frac{9x+bx}{x-3}}{\frac{b+9}{x-3}}=\frac{x(9+b)}{b+9}=x$$ **OK**

OR $$f\left[f^{-1}(x)\right]=f\left[\frac{3x+b}{x-3}\right]=\frac{3\left(\frac{3x+b}{x-3}\right)+b}{\frac{3x+b}{x-3}-3}=\frac{\frac{9x+3b+bx-3b}{x-3}}{\frac{3x+b-3x+9}{x-3}}=\frac{\frac{9x+bx}{x-3}}{\frac{b+9}{x-3}}=\frac{x(9+b)}{b+9}=x$$ **OK**

5. Given that $f(x)=\log_a x$, **find** $f^{-1}(x)$.

Ans:

Initial Function: $f(x)=\log_a x$

Step-1: **Replace f(x) with y:** $y=\log_a x$

Step-2: **Switch x and y:** $x=\log_a y$

Step-3: **Solve for y:** $y=a^x$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=a^x$

Note: $f\left[f^{-1}(x)\right]=f\left(a^x\right)=\log_a a^x=x\log_a a=x$ **OK**

OR $f^{-1}\left[f(x)\right]=f^{-1}\left(\log_a x\right)=a^{\log_a x}=?$

$$z=a^{\log_a x}\Rightarrow\log_a z=\log_a\left(a^{\log_a x}\right)=\log_a x\left(\log_a a\right)=\log_a x$$

$$\Rightarrow\log_a z=\log_a x\Rightarrow z=x$$ **OK**

6. Given that $f(x)=x^2$ **find** $g(x)=f^{-1}(x)$.

Ans:

Initial Function: $f(x)=4x^2+5$

Step-1: **Replace f(x) with y:** $y=4x^2+5$

Step-2: **Switch x and y:** $x=4y^2+5$

Step-3: **Solve for y:** $y=\sqrt{\frac{x-5}{4}}=\frac{\sqrt{x-5}}{2}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\frac{\sqrt{x-5}}{2}$

$$\Rightarrow g(x)=f^{-1}(x)=\frac{\sqrt{x-5}}{2}$$

7. Given that $f(x)=4x+5$, **and** $g(x)=5x-k$. **Find** the value of ***k*** for which $fg=gf$.

Ans:

$$fg=fg(x)=f(5x-k)=4(5x-k)+5=20x-4k+5$$
$$gf=gf(x)=g(4x+5)=5(4x+5)-k=20x+25-k$$
$$fg=gf\Rightarrow 20x-4k+5=20x+25-k\Rightarrow -4k+k=25-5=20\Rightarrow k=-\frac{20}{3}$$

8. Given that $f(x)=\frac{1}{x+2}$, **find** $ff(3)$ **and** $f^{-1}(3)$.

Ans: $ff(3)=f\left(\frac{1}{3+2}\right)=f\left(\frac{1}{5}\right)=\frac{1}{\frac{1}{5}+2}=\frac{1}{\frac{1+10}{5}}=\frac{5}{11}$

Initial Function: $f(x)=\frac{1}{x+2}$

Step-1: **Replace f(x) with y:** $y=\frac{1}{x+2}$

Step-2: **Switch x and y:** $x=\frac{1}{y+2}$

Step-3: **Solve for y:** $y=\frac{1-2x}{x}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\frac{1-2x}{x}$

$$\Rightarrow f^{-1}(3)=\frac{1-6}{3}=-\frac{5}{3}$$

9. If $f(x)=\sqrt{x}$, **and** $g(x)=x+5$, then **express** the following functions in terms of ***f*(x)** and ***g*(x).**

(a) $f_1(x)=\sqrt{x+5}$ (b) $f_2(x)=x-5$ (c) $f_3(x)=x+10$

(d) $f_4(x)=\sqrt{x}+10$ (e) $f_5(x)=x^2+5$

Ans:

(a) $fg=fg(x)=f(x+5)=\sqrt{x+5} \quad \Rightarrow f_1(x)=fg=\sqrt{x+5}$

(b) $g^{-1}(x)=x-5 \quad \Rightarrow f_2(x)=g^{-1}(x)=x-5$

(c) $gg=gg(x)=g(x+5)=(x+5)+5=x+10 \quad \Rightarrow f_3(x)=gg=x+10$

(d) $ggf=ggf(x)=gg\left(\sqrt{x}\right)=g\left(\sqrt{x}+5\right)=\left(\sqrt{x}+5\right)+5=\sqrt{x}+10$

$\Rightarrow f_4(x)=ggf=\sqrt{x}+10$

(e) $gf^{-1}=gf^{-1}(x)=g\left(x^2\right)=x^2+5 \quad \Rightarrow f_5(x)=gf^{-1}(x)=x^2+5$

10. If $f(k)=9^{2k}-5^{2k}$, **where** k is a positive integer

(a) find $f(0)$ and $f(1)$

(b) find $f(k+1)$

(c) Prove that $f(k+1)-25f(k)=56\left(9^{2k}\right)$

(d) Hence, using induction, prove that $f(k)$ is always divisible by 7.

Ans:

(a) $f(k)=9^{2k}-5^{2k}$

$\Rightarrow f(0)=9^0-5^0=1-1=0$ **and** $f(1)=9^2-5^2=81-25=56$

(b) $f(k)=9^{2k}-5^{2k} \quad \Rightarrow f(k+1)=9^{2k+2}-5^{2k+2}=81\left(9^{2k}\right)-25\left(5^{2k}\right)$

(c) $f(k)=9^{2k}-5^{2k}$

$\Rightarrow f(k+1)-25f(k)=81\left(9^{2k}\right)-25\left(5^{2k}\right)-25\left(9^{2k}\right)+25\left(5^{2k}\right)=56\left(9^{2k}\right)$

(d) $f(k)=9^{2k}-5^{2k}$

$f(1)=9^2-5^2=81-25=56$

$f(2)=9^4-5^4=6561-625=5936$

$f(3)=9^6-5^6=531441-15625=515816$

11. If $f(x)=3x-7$ **and** $g(x)=x^2$, **find:**

(a) $fg(x)$ (b) $gf(x)$ (c) $f^{-1}(-4)$

(d) $g^{-1}(x)$ (e) $f^{-1}g^{-1}(x)$

Ans: **(a)** $fg(x)=f\left(x^2\right)=3x^2-7$

(b) $gf(x)=g(3x-7)=(3x-7)^2$

(c)

Initial Function: $f(x)=3x-7$

Step-1: **Replace f(x) with y:** $y=3x-7$

Step-2: **Switch x and y:** $x=3y-7$

Step-3: **Solve for y:** $y=\dfrac{x+7}{3}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\dfrac{x+7}{3}$

$$\Rightarrow f^{-1}(-4)=\frac{-4+7}{3}=-\frac{3}{3}=-1$$

(d)

Initial Function: $g(x)=x^2$

Step-1: **Replace g(x) with y:** $y=x^2$

Step-2: **Switch x and y:** $x=y^2$

Step-3: **Solve for y:** $y=\sqrt{x}$

Step-4: **Replace y with $g^{-1}(x)$:** $g^{-1}(x)=\sqrt{x}$

(e) $f^{-1}g^{-1}(x)=f^{-1}\left(\sqrt{x}\right)=\dfrac{\sqrt{x}+7}{3}$

12. If $f(x)=7x+3$ and $g(x)=x^3$, find:

(a) $fg(x)$ (b) $gf(x)$ (c) $f^{-1}(x)$

(d) $g^{-1}(x)$ (e) $f^{-1}g^{-1}(x)$ (f) $(fg)^{-1}(10)$

Ans: **(a)** $fg(x)=f\left(x^3\right)=7x^3+3$

(b) $gf(x)=g(7x+3)=(7x+3)^3+3$

(c)

Initial Function: $f(x)=7x+3$

Step-1: **Replace f(x) with y:** $y=7x+3$

Step-2: **Switch x and y:** $x=7y+3$

Step-3: **Solve for y:** $y=\dfrac{x-3}{7}$

Step-4: **Replace y with $f^{-1}(x)$:** $f^{-1}(x)=\dfrac{x-3}{7}$

(d)

Initial Function: $g(x)=x^3$

Step-1: **Replace g(x) with y:** $y=x^3$

Step-2: **Switch x and y:** $x=y^3$

Step-3: **Solve for y:** $y=\sqrt[3]{x}$

Step-4: **Replace y with $g^{-1}(x)$:** $g^{-1}(x)=\sqrt[3]{x}$

(e)

$$f^{-1}(x)=\frac{x-3}{7} \quad and \quad g^{-1}(x)=\sqrt[3]{x}$$

$$f^{-1}g^{-1}(x)=f^{-1}\left(\sqrt[3]{x}\right)=\frac{\sqrt[3]{x}-3}{7}$$

(f)

Initial Function: $fg(x) = 7x^3 + 3$

Step-1: **Replace g(x) with y:** $y = 7x^3 + 3$

Step-2: **Switch x and y:** $x = 7y^3 + 3$

Step-3: **Solve for y:** $y = \sqrt[3]{\frac{x-3}{7}}$

Step-4: **Replace y with $(fg)^{-1}(x)$:** $(fg)^{-1}(x) = \sqrt[3]{\frac{x-3}{7}}$

$$\Rightarrow (fg)^{-1}(10) = \sqrt[3]{\frac{10-3}{7}} = 1$$

13. Consider the following functions:

$f(x) = 2x$ $\quad$ $g(x) = x^2$ $\quad$ $h(x) = x - 1$

Find:

(a) fgh (b) fhg (c) gfh

(d) ghf (e) hfg (f) hgf

Ans:

(a) $fgh = fgh(x) = fg(x-1) = f(x-1)^2 = 2(x-1)^2$

(b) $fhg = fhg(x) = fh(x^2) = f(x^2-1) = 2(x^2-1)$

(c) $gfh = gfh(x) = gf(x-1) = g(2x-2) = (2x-2)^2 = 4(x-1)^2$

(d) $ghf = ghf(x) = gh(2x) = g(2x-1) = (2x-1)^2$

(e) $hfg = hfg(x) = hf(x^2) = h(2x^2) = 2x^2 - 1$

(f) $hgf = hgf(x) = hg(2x) = h\left[(2x)^2\right] = h(4x^2) = 4x^2 - 1$

14. Consider the following functions:

$f(x)=2x$ $\quad$ $g(x)=x^2$ $\quad$ $h(x)=x-1$

Find:

(a) $fgh(0)$ (b) $fhg(-2)$ (c) $gfh(5)$

(d) $ghf(a)$ (e) $hfg(b)$ (f) $hgf(k)$

Ans:

(a) $fgh=fgh(x)=fg(x-1)=f(x-1)^2=2(x-1)^2 \Rightarrow fgh(0)=2$

OR $fgh(0)=fg(0-1)=f\left[(-1)^2\right]=f(1)=2(1)=2$

(b) $fhg=fhg(x)=fh(x^2)=f(x^2-1)=2(x^2-1) \Rightarrow fhg(-2)=2(4-1)=6$

OR $fhg(-2)=fh(4)=f(4-1)=f(3)=2(3)=6$

(c) $gfh=gfh(x)=gf(x-1)=g(2x-2)=(2x-2)^2=4(x-1)^2 \Rightarrow ghf(5)=4(5-1)^2=64$

OR $gfh(5)=gf(5-1)=gf(4)=g(8)=(8)^2=64$

(d) $ghf=ghf(x)=gh(2x)=g(2x-1)=(2x-1)^2 \Rightarrow ghf(a)=(2a-1)^2$

OR $ghf(a)=gh(2a)=g(2a-1)=(2a-1)^2$

(e) $hfg=hfg(x)=hf(x^2)=h(2x^2)=2x^2-1 \Rightarrow hfg(b)=2b^2-1$

OR $hfg(b)=hf(b^2)=h(2b^2)=2b^2-1$

(f) $hgf=hgf(x)=hg(2x)=h\left[(2x)^2\right]=h(4x^2)=4x^2-1 \Rightarrow hgf(k)=4k^2-1$

OR $hgf(k)=hg(2k)=h\left[(2k)^2\right]=h(4k^2)=4k^2-1$

15. Consider the following functions, **find** $f(x)$.

(a) $f(x-1)=x^2+x+1$ (b) $f(x+2)=x^2+x-6$

(c) $f(x-3)=x^2-7x+10$ (d) $f\left(\frac{x}{2}\right)=\frac{1}{4}x^2+\frac{3}{2}x+5$

Ans:

(a) $f(x-1)=x^2+x+1=(x-1)^2+3(x-1)x+3$

$\Rightarrow f(x)=x^2+3x+3$

(b) $f(x+2)=x^2+x-6=(x+2)^2-3x-10=(x+2)^2-3(x+2)-4$

$\Rightarrow f(x)=x^2-3x-4$

(c) $f(x-3)=x^2-7x+10=(x-3)^2-x+1=(x-3)^2-(x-3)-2$

$\Rightarrow f(x)=x^2-x-2$

(d) $f\left(\frac{x}{2}\right)=\frac{1}{4}x^2+\frac{3}{2}x+5=\left(\frac{x}{2}\right)^2+3\left(\frac{x}{2}\right)+5$

$\Rightarrow f(x)=x^2+3x+5$

الله(ج) د وکړي چې تاسې ما ته، زما مور او پلار ته، زما اولادونو ته، زما ټولې کورنۍ اوټولو مسلمانانو ته د زړه له کومې دعا کړيې وي، **اوکنه؟** نو اوس يې لطفاً وکړئ!

په پای کې له تاسو ټولو سره له لرې ځايه خدای په اماني کوم۰ له الله(ج) څخه هيله کوم چې په افغانستان او په ټوله نړۍ کې هغه شانته صلح راشي، چې الله(ج) ته مقبوله او منظوره وي، او الله(ج) دې، د افغانانو ترمنځ د حق فيصله وکړي۰ آمين يارب العٰلمين۰