بِسۡمِ اللّٰهِ الرَّحۡمٰنِ الرَّحِيۡمِ

"In the name of Allah ,the Most Beneficent ,the Most Merciful."

شروع کوم په نامه د الله(ج) چې رحمٰن او رحیم دی.

Narrated Abu Qatada (رضي الله عنه): The Prophet (صلى الله عليه وسلم), said, "A true good dream is from Allah (SWT), and a bad dream is from satan."

[H 6984 Al-Bukhari, Arabic-English, Volume 9, By Dr Mohammad Muhsin Khan)]

ګرانو او قدرمنو وروڼو اوخویندو! السلام علیکم ورحمت الله وبرکاته!

له الله(ج) څخه هیله کوم چې تاسې ټول روغ او جوړ وئ. دغه لاندې د ریاضي درسونه به ان شاءالله ځینې پلرونو او په خاصه توګه، د دوی اولادونو ته ډېر ګټور وي.

نوټ: دغه د درسونو لړۍ به ان شاءالله، له لومړي ټولګي څخه تر پوهنتونه پورې دوام ولري. د یادونې وړ ده، چې زما مسلک ریاضي نده، او ما خپلې زده کړې، د دکتورا تر کچې د الکترانکس په برخه کې، په انګلستان کې سرته رسولې. همدارنګه، ما ۶ کاله د کابل پوهنتون د انجینرۍ په پوهنځی، او ۲۷ کاله د انګلستان په یوه پوهنتون کې معلمي کړېده. یعنې هغه چا چې په الکترانکس انجینرنګ کې زده کړې کړې وي، نو هغه د ریاضي په معنٰی ډېر ښه پوهیږي. زه به ان شاءالله د ریاضي له درسونو وروسته، د الکترانکس په برخه کې درسونه شروع کړم.

نوټ: زه د ریاضي د درسونو له ټولو قدرمنو لوستونکو څخه هیله کوم، چې له دغو درسونو څخه اعظمي ګټه پورته کړي. زه د خپل وخت ډېره قیمتي برخه ددغه درسونو په برابرولو کې تېروم. زه هغه وخت ډېر خوشحاله کېږم، چې یو ګران وطنوال له دغو درسونو ګټه پورته کړي. والسلام عبدالله وردک

Dear Brothers and Sisters! السلام علیکم و رحمت الله و برکاته

I kindly request each one of you to remember me, my parents, my children, my family and the entire Muslims of the world in your daily Prayers.

ګرانو او قدرمنو وروڼو اوخویندو! السلام علیکم ورحمت الله وبرکاته!

له الله(ج) څخه هیله کوم چې تاسې ټول روغ او جوړ وئ، او الله(ج) د وکړي، چې ماته ، زما مور او پلار ته، زما اولادونو ته ، زما ټولې کورنۍ ته او د نړۍ ټولو مسلمانانو ته ، د زړه له کومې دعا وکړئ. والسلام

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Chapter-37 Partial Fractions and Binomial Theorem

1. Partial Fractions

Partial fraction is used to split complicated fractions into many simple fractions. For further understanding, consider the following examples.

Example: Express $\frac{2}{x^2-1}$ in partial fractions.

Ans: $\frac{2}{x^2-1}=\frac{2}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$

Method 1:

$$\frac{2}{x^2-1}=\frac{2}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}=\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

So: $2=A(x-1)+B(x+1)$

at $x=-1,\ 2=A(-1-1)+B(-1+1)=-2A \Rightarrow A=-1$

at $x=1,\ 2=A(1-1)+B(1+1)=2B \Rightarrow B=1$

So: $\frac{2}{x^2-1}=\frac{2}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}=-\frac{1}{x+1}+\frac{1}{x-1}$

Method 2:

$$\frac{2}{x^2-1}=\frac{2}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$$

To find **A**, **remove** the **denominator** of **A** from the original fraction, and then evaluate the remaining fraction at the root of denominator **A**, **i.e.**, in this case at **x = -1**

Therefore, $A=\left.\frac{2}{x-1}\right|_{x=-1}=\frac{2}{-1-1}=-\frac{2}{2}=-1$

To find **B**, **remove** the **denominator** of **B** from the original fraction, and then evaluate the remaining fraction at the root of denominator **B**, **i.e.**, in this case at **x = 1**

Hence, $B=\left.\frac{2}{x+1}\right|_{x=1}=\frac{2}{1+1}=\frac{2}{2}=1$

So: $\frac{2}{x^2-1}=\frac{2}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}=-\frac{1}{x+1}+\frac{1}{x-1}$

Verify: $\frac{2}{x^2-1}\overset{?}{=}-\frac{1}{x+1}+\frac{1}{x-1}=\frac{-(x-1)+(x+1)}{(x+1)(x-1)}=\frac{-x+1+x+1}{x^2-1}=\frac{2}{x^2-1}\Rightarrow$ **OK**

Example: Express $\frac{3x-1}{(x+1)(x-3)}$ in partial fractions.

Ans: $$\frac{3x-1}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}$$

Method 1:

$$\frac{3x-1}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}=\frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

So: $3x-1=A(x-3)+B(x+1)$

at $x=-1$, $3(-1)-1=A(-1-3)+B(-1+1)=-4A=-4\Rightarrow A=1$

at $x=3$, $3(3)-1=A(3-3)+B(3+1)=4B=9-1=8\Rightarrow B=2$

So: $$\frac{3x-1}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}=\frac{1}{x+1}+\frac{2}{x-3}$$

Method 2:

$$\frac{3x-1}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}$$

To find **A**, **remove** the **denominator** of **A** from the original fraction, and then evaluate the remaining fraction at the root of denominator **A**, **i.e.**, in this case at **x = -1**

Therefore, $$A=\left.\frac{3x-1}{x-3}\right|_{x=-1}=\frac{-3-1}{-1-3}=\frac{-4}{-4}=1$$

To find **B**, **remove** the **denominator** of **B** from the original fraction, and then evaluate the remaining fraction at the root of denominator **B**, **i.e.**, in this case at **x = 3**

and $$B=\left.\frac{3x-1}{x+1}\right|_{x=3}=\frac{9-1}{3+1}=\frac{8}{4}=2$$

So: $$\frac{3x-1}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}=\frac{1}{x+1}+\frac{2}{x-3}$$

Verify: $$\frac{3x-1}{(x+1)(x-3)}\overset{?}{=}\frac{1}{x+1}+\frac{2}{x-3}=\frac{(x-3)+2(x+1)}{(x+1)(x-3)}=\frac{x-3+2x+2}{(x+1)(x-3)}=\frac{3x-1}{(x+1)(x-3)}$$

Example: Express $\frac{x+1}{x^3+2x^2-8x}$ in partial fractions.

Ans: $$\frac{x+1}{x^3+2x^2-8x}=\frac{x+1}{x(x^2+2x-8)}=\frac{x+1}{x(x+4)(x-2)}=\frac{A}{x}+\frac{B}{x+4}+\frac{C}{x-2}$$

Method 1:

$$\frac{x+1}{x^3+2x^2-8x}=\frac{x+1}{x(x^2+2x-8)}=\frac{x+1}{x(x+4)(x-2)}=\frac{A}{x}+\frac{B}{x+4}+\frac{C}{x-2}=\frac{A(x+4)(x-2)+Bx(x-2)+Cx(x+4)}{x(x+4)(x-2)}$$

So: $x+1=A(x+4)(x-2)+Bx(x-2)+Cx(x+4)$

at $x = 0,\ 0+1 = A(0+4)(0-2) + B(0)(0-2) + C(0)(0+4)$

$-8A = 1 \Rightarrow A = -\frac{1}{8}$

at $x = -4,\ -4+1 = A(-4+4)(-4-2) + B(-4)(-4-2) + C(-4)(-4+4)$

$$-3 = A(0)(-4-2) + B(-4)(-4-2) + C(-4)(0) \Rightarrow -24B = -3 \Rightarrow B = \frac{1}{8}$$

at $x = 2,\ 2+1 = A(2+4)(2-2) + B(2)(2-2) + C(2)(2+4)$

$3 = A(6)(0) + B(2)(0) + 12C \Rightarrow C = \frac{3}{12} = \frac{1}{4}$

So: $$\frac{x+1}{x^3+2x^2-8x} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-2} = -\frac{1}{8x} + \frac{1}{8(x+4)} + \frac{1}{4(x-2)}$$

Method 2:

$$\frac{x+1}{x^3+2x^2-8x} = \frac{x+1}{x(x^2+2x-8)} = \frac{x+1}{x(x+4)(x-2)} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-2}$$

To find **A**, **remove** the **denominator** of **A** from the original fraction, and then evaluate the remaining fraction at the root of denominator **A**, **i.e.**, in this case at **x = 0**

$$A = \left.\frac{x+1}{(x+4)(x-2)}\right|_{x=0} = \frac{0+1}{(0+4)(0-2)} = -\frac{1}{8}$$

To find **B**, **remove** the **denominator** of **B** from the original fraction, and then evaluate the remaining fraction at the root of denominator **B**, **i.e.**, in this case at **x = -4**

$$B = \left.\frac{x+1}{x(x-2)}\right|_{x=-4} = \frac{-4+1}{-4(-4-2)} = -\frac{3}{24} = -\frac{1}{8}$$

To find **C**, **remove** the **denominator** of **C** from the original fraction, and then evaluate the remaining fraction at the root of denominator **C**, **i.e.**, in this case at **x = 2**

and $$C = \left.\frac{x+1}{x(x+4)}\right|_{x=2} = \frac{2+1}{2(2+4)} = \frac{3}{12} = \frac{1}{4}$$

So: $$\frac{x+1}{x^3+2x^2-8x} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-2} = -\frac{1}{8x} - \frac{1}{8(x+4)} + \frac{1}{4(x-2)}$$

Verify: $$\frac{x+1}{x(x+4)(x-2)} \stackrel{?}{=} -\frac{1}{8x} - \frac{1}{8(x+4)} + \frac{1}{4(x-2)} = \frac{-(x+4)(x-2) - x(x-2) + 2x(x+4)}{8x(x+4)(x-2)} = \frac{x+1}{x(x+4)(x-2)}$$

Example: Express $\dfrac{8x^2+11x-5}{(x+3)(x^2+3x-18)}$ in partial fractions.

Ans: $$\frac{8x^2+11x-5}{(x+2)(x^2+3x-18)}=\frac{8x^2+11x-5}{(x+2)(x+6)(x-3)}=\frac{A}{x+2}+\frac{B}{x+6}+\frac{C}{x-3}$$

Method 1:

$$\frac{8x^2+11x-5}{(x+2)(x^2+3x-18)}=\frac{8x^2+11x-5}{(x+2)(x+6)(x-3)}=\frac{A}{x+2}+\frac{B}{x+6}+\frac{C}{x-3}=\frac{A(x+6)(x-3)+B(x+2)(x-3)+C(x+2)(x+6)}{(x+2)(x+6)(x-3)}$$

$$\therefore 8x^2+11x-5=A(x+6)(x-3)+B(x+2)(x-3)+C(x+2)(x+6)$$

at $x=-6$,

$$8(-6)^2+11(-6)-5=A(-6+6)(-6-3)+B(-6+2)(-6-3)+C(-6+2)(-6+6)$$

$$288-66-5=0+36B+0\Rightarrow 36B=217\Rightarrow B=\frac{217}{36}$$

at $x=-2$,

$$8(-2)^2+11(-2)-5=A(-2+6)(-2-3)+B(-2+2)(-2-3)+C(-2+2)(-2+6)$$

$$32-22-5=-20A+0+0\Rightarrow -20A=5\Rightarrow A=\frac{5}{-20}=-\frac{1}{4}$$

at $x=3$,

$$8(3)^2+11(3)-5=A(3+6)(3-3)+B(3+2)(3-3)+C(3+2)(3+6)$$

$$72+33-5=0+0+45C\Rightarrow 45C=105-5=100\Rightarrow C=\frac{100}{45}=\frac{20}{9}$$

So: $$\frac{8x^2+11x-5}{(x+2)(x^2+3x-18)}=\frac{8x^2+11x-5}{(x+2)(x+6)(x-3)}=\frac{A}{x+2}+\frac{B}{x+6}+\frac{C}{x-3}$$

$$=-\frac{1}{4(x+2)}+\frac{217}{36(x+6)}+\frac{20}{9(x-3)}$$

Method 2:

$$\frac{8x^2+11x-5}{(x+2)(x^2+3x-18)}=\frac{8x^2+11x-5}{(x+2)(x+6)(x-3)}=\frac{A}{x+2}+\frac{B}{x+6}+\frac{C}{x-3}$$

To find **A**, **remove** the **denominator** of **A** from the original fraction, and then evaluate the remaining fraction at the root of denominator **A**, **i.e.**, in this case at **x = -2**

Therefore,

$$A=\left.\frac{8x^2+11x-5}{(x+6)(x-3)}\right|_{x=-2}=\frac{8(-2)^2+11(-2)-5}{(-2+6)(-2-3)}=\frac{32-22-5}{4(-5)}=\frac{5}{-20}=-\frac{1}{4}$$

To find **B**, **remove** the **denominator** of **B** from the original fraction, and then evaluate the remaining fraction at the root of denominator **B**, **i.e.**, in this case at **x = -6**

$$B=\left.\frac{8x^2+11x-5}{(x+2)(x-3)}\right|_{x=-6}=\frac{8(-6)^2+11(-6)-5}{(-6+2)(-6-3)}=\frac{288-66-5}{-4(-9)}=\frac{217}{36}$$

To find **C**, **remove** the **denominator** of **C** from the original fraction, and then evaluate the remaining fraction at the root of denominator **C**, **i.e.**, in this case at **x = 3**

$$C = \left.\frac{8x^2 + 11x - 5}{(x+2)(x+6)}\right|_{x=3} = \frac{8(3)^2 + 11(3) - 5}{(3+2)(3+6)} = \frac{72 + 33 - 5}{45} = \frac{100}{45} = \frac{20}{9}$$

So: $$\frac{8x^2 + 11x - 5}{(x+2)(x^2 + 3x - 18)} = \frac{8x^2 + 11x - 5}{(x+2)(x+6)(x-3)} = \frac{A}{x+2} + \frac{B}{x+6} + \frac{C}{x-3}$$

$$= -\frac{1}{4(x+2)} + \frac{217}{36(x+6)} + \frac{20}{9(x-3)}$$

Example: Express $\frac{2x-5}{(x-4)^2}$ in partial fractions.

Ans: $$\frac{2x-5}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$$

Method 1:

$$\frac{2x-5}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2} = \frac{A(x-4) + B}{(x-4)^2}$$

\ $2x - 5 = A(x-4) + B = Ax - 4A + B$

Þ $2x - 5 = Ax - 4A + B$

By comparison: A = 2 (**i.e.** the coefficient of **x**) **and**

$-4A + B = -5$ Þ $B = -5 + 4A = -5 + 8 = 3$ Þ $B = 3$

So: $$\frac{2x-5}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2} = \frac{2}{(x-4)} + \frac{3}{(x-4)^2}$$

Method 2:

$$\frac{2x-5}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$$

To find **B**, remove the denominator of **B** from the original fraction, and then evaluate the remaining fraction at the repeated root of denominator **B**, i.e., in this case at **x = 4**

Therefore, $B = \left.2x - 5\right|_{x=4} = 2(4) - 5 = 8 - 5 = 3$

To find **A**, remove the denominator of **B** from the original fraction, and then take the derivative of the remaining fraction $(i.e.\ 2x - 5)$ and evaluate that at the repeated root of the denominator **B**, i.e., in this case at **x = 4**

$$\therefore A = \frac{1}{1!} \times \frac{d}{dx}(2x - 5)\Big|_{x=4} = 2$$

So: $$\frac{2x-5}{(x-4)^2}=\frac{A}{x-4}+\frac{B}{(x-4)^2}=\frac{2}{x-4}+\frac{3}{(x-4)^2}$$

Example: Express $\frac{3x-2}{(x-5)^3}$ in partial fractions.

Ans: $$\frac{3x-2}{(x-5)^3}=\frac{A}{x-5}+\frac{B}{(x-5)^2}+\frac{C}{(x-5)^3}$$

Method 1:

$$\frac{3x-2}{(x-5)^3}=\frac{A}{x-5}+\frac{B}{(x-5)^2}+\frac{C}{(x-5)^3}=\frac{A(x-5)^2+B(x-5)+C}{(x-5)^3}$$

So: $3x-2=A(x-5)^2+B(x-5)+C$

$\Rightarrow 3x-2=A(x^2-10x+25)+Bx-5B+C=Ax^2-10Ax+25A+Bx-5B+C$

$\Rightarrow 3x-2=Ax^2-(10A-B)x+25A-5B+C$

By comparison: A = 0, and $-10A+B=3 \Rightarrow 0+B=3 \Rightarrow B=3$

and $25A-5B+C=-2 \Rightarrow 0-5(3)+C=-2 \Rightarrow C=15-2=13$

So: $$\frac{3x-2}{(x-5)^3}=\frac{A}{x-5}+\frac{B}{(x-5)^2}+\frac{C}{(x-5)^3}=\frac{0}{x-5}+\frac{3}{(x-5)^2}+\frac{13}{(x-5)^3}$$

Method 2:

$$\frac{3x-2}{(x-5)^3}=\frac{A}{x-5}+\frac{B}{(x-5)^2}+\frac{C}{(x-5)^3}$$

To find **C**, remove the denominator of **C** from the original fraction, and then evaluate the remaining fraction at the repeated root of denominator **C**, i.e. at **x = 5**

Therefore, $C=3x-2\Big|_{x=5}=3(5)-2=15-2=13$

To find **B**, remove the denominator of **C** from the original fraction, and then take the **first derivative** of the remaining fraction and evaluate that at the repeated root of the denominator of **B**, i.e. at **x = 5**

and $B=\frac{1}{1!}\frac{d}{dx}(3x-2)\Big|_{x=5}=3$

To find **A**, remove the denominator of **C** from the original fraction, and then take the **2nd derivative** of the remaining fraction and evaluate at the root of denominator **A**, i.e. at **x = 5**

and $A=\frac{1}{2!}\times\frac{d^2}{dx^2}(3x-5)\Big|_{x=5}=\frac{0}{2}=0$

So: $\dfrac{3x-2}{(x-5)^3} = \dfrac{A}{x-5} + \dfrac{B}{(x-5)^2} + \dfrac{C}{(x-5)^3} = \dfrac{0}{x-5} + \dfrac{3}{(x-5)^2} + \dfrac{13}{(x-5)^3}$

$\therefore \dfrac{3x-2}{(x-5)^3} = \dfrac{3}{(x-5)^2} + \dfrac{13}{(x-5)^3} = \dfrac{3(x-5)+13}{(x-5)^3} = \dfrac{3x-15+13}{(x-5)^3} = \dfrac{3x-2}{(x-5)^3}$ **OK**

Example: Express $\dfrac{2x-3}{(x-2)^4}$ in partial fractions.

Ans: $\dfrac{2x-3}{(x-2)^4} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3} + \dfrac{D}{(x-2)^4}$

Method 1:

$$\frac{2x-3}{(x-2)^4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4} = \frac{A(x-2)^3 + B(x-2)^2 + C(x-2) + D}{(x-2)^3}$$

So: $2x-3 = A(x-2)^3 + B(x-2)^2 + C(x-2) + D$

$\Rightarrow 2x-3 = Ax^3 - 6Ax^2 + 12Ax - 8A + Bx^2 - 4Bx + 4B + Cx - 2C + D$

By comparison: A = 0, and $-6A + B = 0 \Rightarrow 0 + B = 0 \Rightarrow B = 0$

and $12A - 4B + C = 2 \Rightarrow 0 - 0 + C = 2 \Rightarrow C = 2$

and $-8A + 4B - 2C + D = -3 \Rightarrow 0 + 0 - 4 + D = -3 \Rightarrow D = -3 + 4 = 1$

So: $\dfrac{2x-3}{(x-2)^4} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3} + \dfrac{D}{(x-2)^4} = \dfrac{0}{x-2} + \dfrac{0}{(x-2)^2} + \dfrac{2}{(x-2)^3} + \dfrac{1}{(x-2)^4}$

$\backslash \dfrac{2x-3}{(x-2)^4} = \dfrac{2}{(x-2)^3} + \dfrac{1}{(x-2)^4} = \dfrac{2(x-2)+1}{(x-2)^4} = \dfrac{2x-4+1}{(x-2)^4} = \dfrac{2x-3}{(x-2)^4}$ **OK**

Method 2:

$$\frac{2x-3}{(x-2)^4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4}$$

To find **D**, remove the denominator of **D** from the original fraction, and then evaluate the remaining fraction at the repeated root of denominator **D**, i.e., in this case at **x = 2**

Therefore, $D = 2x-3\Big|_{x=2} = 3(2) - 3 = 4 - 3 = 1$

To find **C**, remove the denominator of **D** from the original fraction, and then take the **1st derivative** of the remaining fraction and evaluate that at the repeated root of the denominator of **C**, i.e. at **x = 2**

and $C = \dfrac{1}{1!}\dfrac{d}{dx}(2x-3)\Big|_{x=2} = 2$

To find **B**, remove the denominator of **C** from the original fraction, and then take the **2nd derivative** of the remaining fraction and evaluate at the root of denominator **B**, i.e. at **x = 2**

and $B = \frac{1}{2!} \times \frac{d^2}{dx^2}(2x-3)\Big|_{x=2} = \frac{0}{2} = 0$

To find **A**, remove the denominator of **C** from the original fraction, and then take the **3rd derivative** of the remaining fraction and evaluate at the root of denominator **A**, i.e. at **x = 2**

and $B = \frac{1}{3!} \times \frac{d^3}{dx^3}(2x-3)\Big|_{x=2} = \frac{0}{6} = 0$

So: $\frac{2x-3}{(x-2)^4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4} = \frac{0}{x-2} + \frac{0}{(x-2)^2} + \frac{2}{(x-2)^3} + \frac{1}{(x-2)^4}$

$\therefore \frac{2x-3}{(x-2)^4} = \frac{2}{(x-2)^3} + \frac{1}{(x-2)^4} = \frac{2(x-2)+1}{(x-2)^4} = \frac{2x-4+1}{(x-2)^4} = \frac{2x-3}{(x-2)^4}$ **OK**

Example: Express $\frac{x^2-3x-2}{(x-2)^3}$ in partial fractions.

Ans: $\frac{x^2-3x-2}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$

Method 1:

$$\frac{x^2-3x-2}{(x-2)^3} = \frac{A(x-2)^2 + B(x-2) + C}{(x-2)^3} = \frac{A(x^2-4x+4) + Bx - 2B + C}{(x-2)^3}$$

So: $x^2-3x-2 = A(x^2-4x+4) + Bx - 2B + C$

$$x^2-3x-2 = Ax^2 - 4Ax + 4A + Bx - 2B + C = Ax^2 - (4A-B)x + 4A - 2B + C$$

By comparison: A = 1 (coefficient of x^2)

and $-(4A-B) = -3 \Rightarrow B = -3 + 4A = -3 + 4 = 1$ (coefficient of x)

and $4A - 2B + C = -2 \Rightarrow C = -2 - 4A + 2B = -2 - 4 + 2 = -4$ (Const term)

So: $\frac{x^2-3x-2}{(x-2)^3} = \frac{1}{(x-2)} + \frac{1}{(x-2)^2} - \frac{4}{(x-2)^3}$ **OK**

Method 2:

$$\frac{x^2-3x-2}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

To find **C**, remove the denominator of **C** from the original fraction, and then evaluate the remaining fraction at the repeated root of denominator **C**, i.e., in this case at **x = 2**

Therefore, $C = x^2 - 3x - 2\Big|_{x=2} = 4 - 6 - 2 = -4$

To find **B**, remove the denominator of **C** from the original fraction, and then take the **1st derivative** of the remaining fraction and evaluate that at the repeated root of the denominator of **B**, i.e. at **x = 2**

and $B = \frac{1}{1!}\frac{d}{dx}(x^2 - 3x - 2)\Big|_{x=2} = (2x - 3)\Big|_{x=2} = 4 - 3 = 1$

To find **A**, remove the denominator of **C** from the original fraction, and then take the **2nd derivative** of the remaining fraction and evaluate at the root of denominator **A**, i.e. at **x = 2**

and $C = \frac{1}{2!} \times \frac{d^2}{dx^2}(x^2 - 3x - 2)\Big|_{x=2} = \frac{1}{2!}(2) = \frac{2}{2} = 1$

So: $$\frac{x^2 - 3x - 2}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} = \frac{1}{(x-2)} + \frac{1}{(x-2)^2} - \frac{4}{(x-2)^3}$$

Example: Express $\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)}$ in partial fractions.

Ans: $$\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)} = A + \frac{B}{(x-5)} + \frac{C}{(x+2)} + \frac{D}{(x-3)}$$

Method 1:

$$\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)} = A + \frac{B}{(x-5)} + \frac{C}{(x+2)} + \frac{D}{(x-3)}$$

$$= \frac{A(x-5)(x+2)(x-3) + B(x+2)(x-3) + C(x-5)(x-3) + D(x-5)(x+2)}{(x-5)(x+2)(x-3)}$$

So: $2x^3 + 3x^2 - 15x - 7 = A(x-5)(x+2)(x-3) + B(x+2)(x-3) + C(x-5)(x-3) + D(x-5)(x+2)$

By comparison: A = 2, because of coefficient of x^3

at x = -2, $2(-2)^3 + 3(-2)^2 - 15(-2) - 7 = C(-2-5)(-2-3)$

$-16 + 12 + 30 - 7 = 35C \Rightarrow C = \frac{19}{35}$

at x = 3, $2(3)^3 + 3(3)^2 - 15(3) - 7 = D(3-5)(3+2)$

$54 + 27 - 45 - 7 = -10D \Rightarrow D = -\frac{29}{10}$

at x = 5, $2(5)^3 + 3(5)^2 - 15(5) - 7 = B(5+2)(5-3)$

$250 + 75 - 75 - 7 = 14B \Rightarrow B = \frac{243}{14}$

So: $$\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)} = A + \frac{B}{(x-5)} + \frac{C}{(x+2)} + \frac{D}{(x-3)}$$

$$\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)} = 2 + \frac{243}{14(x-5)} + \frac{19}{35(x+2)} - \frac{29}{10(x-3)}$$

Method 2:

Note: Since both numerator and denominator of the original fraction is of the same order, then the original fraction is divided, where the quotient is **2** and the remainder is: $15x^2 - 13x - 67$.

Long Division

$$
\begin{array}{r|l}
 & \quad 2 \\
\hline
x^3 - 6x^2 - x + 30 & 2x^3 + 3x^2 - 15x - 7 \\
 & \underline{-2x^3 + 12x^2 + 2x - 60} \\
 & \qquad 15x^2 - 13x - 67 \rightarrow \text{remainder}
\end{array}
$$

$$\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)} = 2 + \frac{15x^2 - 13x - 67}{(x-5)(x+2)(x-3)} = 2 + \frac{B}{(x-5)} + \frac{C}{(x+2)} + \frac{D}{(x-3)}$$

$$\Rightarrow \frac{15x^2 - 13x - 67}{(x-5)(x+2)(x-3)} = \frac{B}{(x-5)} + \frac{C}{(x+2)} + \frac{D}{(x-3)}$$

To find **B**, remove the denominator of **B** from $\frac{15x^2 - 13x - 67}{(x-5)(x+2)(x-3)}$, and evaluate the remaining fraction $\frac{15x^2 - 13x - 67}{(x+2)(x-3)}$ at the root of denominator of **B**, i.e. at **x = 5**

Therefore,

$$B = \left.\frac{15x^2 - 13x - 67}{(x+2)(x-3)}\right|_{x=5} = \frac{15(5)^2 - 13(5) - 67}{(5+2)(5-3)} = \frac{375 - 65 - 67}{(7)(2)} = \frac{243}{14}$$

To find **C**, remove the denominator of **C** from $\frac{15x^2 - 13x - 67}{(x-5)(x+2)(x-3)}$, and evaluate the remaining fraction $\frac{15x^2 - 13x - 67}{(x-5)(x-3)}$ at the roots of the denominator of **C**, i.e. at **x = -2**

Therefore,

$$C = \left.\frac{15x^2 - 13x - 67}{(x-5)(x-3)}\right|_{x=-2} = \frac{15(-2)^2 - 13(-2) - 67}{(-2-5)(-2-3)} = \frac{60 + 26 - 67}{(-7)(-5)} = \frac{19}{35}$$

To find **D**, remove the denominator of **D** from $\frac{15x^2 - 13x - 67}{(x-5)(x+2)(x-3)}$, and evaluate the remaining fraction $\frac{15x^2 - 13x - 67}{(x-5)(x+2)}$ at the root of denominator **D**, i.e. at **x = 3**

Therefore,

$$D = \left.\frac{15x^2 - 13x - 67}{(x-5)(x+2)}\right|_{x=3} = \frac{15(3)^2 - 13(3) - 67}{(3-5)(3+2)} = \frac{135 - 39 - 67}{(-2)(5)} = -\frac{29}{10}$$

So: $$\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)} = 2 + \frac{15x^2 - 13x - 67}{(x-5)(x+2)(x-3)} = 2 + \frac{B}{(x-5)} + \frac{C}{(x+2)} + \frac{D}{(x-3)}$$

$$\frac{2x^3 + 3x^2 - 15x - 7}{(x-5)(x+2)(x-3)} = 2 + \frac{243}{14(x-5)} + \frac{19}{35(x+2)} - \frac{29}{10(x-3)}$$

Example: Express $\frac{1}{(x+1)(x^2+2x+2)}$ in partial fractions.

Ans: $$\frac{1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$$

$$A = \left.\frac{1}{x^2+2x+2}\right|_{x=-1} = 1$$

$$\therefore \frac{1}{(x+1)(x^2+2x+2)} = \frac{1}{x+1} + \frac{Bx+C}{x^2+2x+2} = \frac{x^2+2x+2+(x+1)(Bx+C)}{(x+1)(x^2+2x+2)}$$

$$\Rightarrow x^2 + 2x + 2 + Bx^2 + Cx + Bx + C = 1 \qquad \Rightarrow 2 + C = 1 \qquad \Rightarrow C = -1$$

and $2 + C + B = 0$ Þ $B = -C - 2 = 1 - 2 = -1$

$$\therefore \frac{1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2} = \frac{1}{x+1} - \frac{x+1}{x^2+2x+2}$$

Example: Express $\frac{x+1}{x(x+2)(x+3)(x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+3} + \frac{D}{x+4}$

Ans: $$\frac{x+1}{x(x+2)(x+3)(x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+3} + \frac{D}{x+4}$$

$$A = \left.\frac{x+1}{(x+2)(x+3)(x+4)}\right|_{x=0} = \frac{1}{24}$$

$$B = \left.\frac{x+1}{x(x+3)(x+4)}\right|_{x=-2} = \frac{-1}{-2(-2+3)(-2+4)} = \frac{-1}{-4} = \frac{1}{4}$$

$$C = \left.\frac{x+1}{x(x+2)(x+4)}\right|_{x=-3} = \frac{-3+1}{-3(-3+2)(-3+4)} = \frac{-2}{3} = -\frac{2}{3}$$

$$D = \left.\frac{x+1}{x(x+2)(x+3)}\right|_{x=-4} = \frac{-4+1}{-4(-4+2)(-4+3)} = \frac{-3}{-8} = \frac{3}{8}$$

$$\backslash\ \frac{x+1}{x(x+2)(x+3)(x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+3} + \frac{D}{x+4} = \frac{1}{24x} + \frac{1}{4(x+2)} - \frac{2}{3(x+3)} + \frac{3}{8(x+4)}$$

Example: Express $\dfrac{3x^2+5x-8}{(x+2)(x+3)(x+4)}$ as partial fractions.

Ans: $\dfrac{3x^2+5x-8}{(x+2)(x+3)(x+4)} = \dfrac{A}{x+2} + \dfrac{B}{x+3} + \dfrac{C}{x+4}$, **where**

$$A = \left.\frac{3x^2+5x-8}{(x+3)(x+4)}\right|_{x=-2} = \frac{3(-2)^2+5(-2)-8}{(-2+3)(-2+4)} = \frac{-6}{2} = -3$$

$$B = \left.\frac{3x^2+5x-8}{(x+2)(x+4)}\right|_{x=-3} = \frac{3(-3)^2+5(-3)-8}{(-3+2)(-3+4)} = \frac{4}{-1} = -4$$

$$C = \left.\frac{3x^2+5x-8}{(x+2)(x+3)}\right|_{x=-4} = \frac{3(-4)^2+5(-4)-8}{(-4+2)(-4+3)} = \frac{20}{2} = 10$$

$$\frac{3x^2+5x-8}{(x+2)(x+3)(x+4)} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{x+4} = -\frac{3}{x+2} - \frac{4}{x+3} + \frac{10}{x+4}$$ **OK**

2. Binomial Theorem

Example: Consider the following multiplications:

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2.$$

$$(x-y)^2 = (x-y)(x-y) = x^2 - xy - yx + y^2 = x^2 - 2xy + y^2$$

Note: The **number** of **terms** in the **expansion** is **(n + 1) = (2 + 1) = 3**

$$(x+y)^3 = (x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3=(x-y)(x-y)(x-y)=x^3-3x^2y+3xy^2-y^3$$

Note: The **number** of **terms** in the **expansion** is **(n + 1) = (3 + 1) = 4**

Note: $$(x+y)^n=\frac{x^n}{0!}+\frac{nx^{n-1}y}{1!}+\frac{n(n-1)x^{n-2}y^2}{2!}+\frac{n(n-1)(n-2)x^{n-3}y^3}{3!}+\ldots+y^n$$

Provided n is a positive integer!

Note: The **number** of **terms** in the **expansion** is **(n + 1)**.

Example: Calculate ${}^6C_3$.

Note: $${}^nC_r=\frac{n!}{(n-r)!\,r!}$$

Ans: $${}^6C_3=\frac{6!}{(6-3)!3!}=\frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times3\times2\times1}=\frac{720}{36}=20$$

Example: Find the value of ${}^8C_3$.

Ans: $${}^8C_3=\frac{8!}{(8-3)!\,3!}=\frac{8\times7\times6\times5!}{5!3!}=56$$

Note: $$\binom{C}{A}=\frac{C!}{(C-A)!\,A!}$$

Example: $$\binom{4}{2}=\frac{4!}{(4-2)!\,2!}=\frac{4!}{2!2!}=\frac{4\times3\times2\times1}{(2\times1)(2\times1)}=6$$ and $0!=1$

Example: Find the binomial expansion of $(2-x)^3$.

Ans: $$(a+b)^n=\frac{a^n}{0!}+\frac{n\times a^{n-1}b}{1!}+\frac{n(n-1)\times a^{n-2}b^2}{2!}+\frac{n(n-1)(n-2)\times a^{n-3}b^3}{3!}+\ldots+b^n$$

$$\Rightarrow(2-x)^3=\frac{2^3}{0!}+\frac{3\times2^2(-x)}{1!}+\frac{2\times3\times2(-x)^2}{2!}+\frac{1\times2\times3\times2^0(-x)^3}{3!}$$
$$=8-12x+6x^2-x^3$$

OR use the following method, when n is a positive integer!

$$(a-b)^3=\binom{3}{0}a^3-\binom{3}{1}a^2b+\binom{3}{2}ab^2-\binom{3}{3}b^3$$

So: $$(2-x)^3=\binom{3}{0}2^3-\binom{3}{1}2^2x+\binom{3}{2}2x^2-\binom{3}{3}x^3$$

$$\binom{3}{0}=\frac{3!}{(3-0)!\,0!}=\frac{3!}{3!\times 0!}=1 \qquad \binom{3}{1}=\frac{3!}{(3-1)!\,1!}=\frac{3!}{2!\times 1!}=\frac{3\times 2\times 1}{2\times 1\times 1}=\frac{6}{2}=3$$

$$\binom{3}{2}=\frac{3!}{(3-2)!\,2!}=\frac{3!}{1!\times 2!}=3 \qquad \binom{3}{3}=\frac{3!}{(3-3)!\,3!}=\frac{3!}{0!\times 3!}=\frac{3\times 2\times 1}{1\times 3\times 2\times 1}=\frac{6}{6}=1$$

$$\Rightarrow (2-x)^3=2^3-3\times 2^2x+3\times 2x^2-x^3=8-12x+6x^2-x^3$$

Example: Find the binomial expansion of $(2+x)^4$, writing each term as simply as possible.

Ans: $(a+b)^n=\frac{a^n}{0!}+\frac{n\times a^{n-1}b}{1!}+\frac{n(n-1)\times a^{n-2}b^2}{2!}+\frac{n(n-1)(n-2)\times a^{n-3}b^3}{3!}+\ldots+b^n$

$$(2+x)^4=\frac{2^4}{0!}+\frac{4\times 2^3x}{1!}+\frac{4(3)\times 2^2x^2}{2!}+\frac{4(3)(2)\times 2^1x^3}{3!}+\frac{4(3)(2)x^4}{4!}$$

$$(2+x)^4=16+32x+24x^2+8x^3+x^4$$

OR use the following method, when n is a positive integer!

$$(a+b)^4=\binom{4}{0}a^4+\binom{4}{1}a^3b+\binom{4}{2}a^2b^2+\binom{4}{3}ab^3+\binom{4}{4}b^4$$

$$(2+x)^4=\binom{4}{0}2^4+\binom{4}{1}2^3x+\binom{4}{2}2^2x^2+\binom{4}{3}2x^3+\binom{4}{4}x^4$$

$$\binom{4}{0}=\frac{4!}{(4-0)!\,0!}=\frac{4!}{4!\times 0!}=1 \qquad \binom{4}{1}=\frac{4!}{(4-1)!\,1!}=\frac{4!}{3!\times 1!}=4$$

$$\binom{4}{2}=\frac{4!}{(4-2)!\,2!}=\frac{4!}{2!\times 2!}=\frac{4\times 3\times 2\times 1}{2\times 1\times 2\times 1}=\frac{24}{4}=6$$

$$\binom{4}{3}=\frac{4!}{(4-3)!\,3!}=\frac{4!}{01!\times 3!}=4 \qquad \binom{4}{4}=\frac{4!}{(4-4)!\,4!}=\frac{4!}{01!\times 4!}=1$$

$$(2+x)^4=\binom{4}{0}2^4+\binom{4}{1}2^3x+\binom{4}{2}2^2x^2+\binom{4}{3}2x^3+\binom{4}{4}x^4$$

$$(2+x)^4 = 2^4 + 4\times 2^3 x + 6\times 2^2 x^2 + 4\times 2x^3 + x^4$$

$$(2+x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4$$

Example: (a) Find the binomial expansion of $(1+x)^7$ as far as the term in x^5.

(b) Using the result of (a), find the value of $(1.01)^7$ correct to **7 d.p.**

(c) Using the result of (a), find the expansion of $(1-2x)^7$ as far as the term in x^5.

Ans:

(a) $(a+b)^n = \frac{a^n}{0!} + \frac{n\times a^{n-1}b}{1!} + \frac{n(n-1)\times a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)\times a^{n-3}b^3}{3!} + \ldots + b^n$

$$(1+x)^7 = \frac{1^7}{0!} + \frac{7\times 1^6 x}{1!} + \frac{7(6)\times 1^5 x^2}{2!} + \frac{5(7)(6)\times 1^4 x^3}{3!} + \frac{4(5)(7)(6)\times 1^3 x^4}{4!} + \frac{3(4)(5)(7)(6)\times 1^2 x^5}{5!} + \ldots$$

$$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + \ldots$$

\ $(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + \ldots$

OR use the following method, when n is a positive integer!

$$(a+b)^7 = \binom{7}{0}a^7 + \binom{7}{1}a^6 b + \binom{7}{2}a^5 b^2 + \binom{7}{3}a^4 b^3 + \binom{7}{4}a^3 b^4 + \binom{7}{5}a^2 b^5 + \ldots$$

$$(1+x)^7 = \binom{7}{0}1^7 + \binom{7}{1}1^6 x + \binom{7}{2}1^5 x^2 + \binom{7}{3}1^4 x^3 + \binom{7}{4}1^3 x^4 + \binom{7}{5}1^2 x^5 + \ldots$$

$$\binom{7}{0} = \frac{7!}{(7-0)!\,0!} = \frac{7!}{7!\times 0!} = 1 \qquad \binom{7}{1} = \frac{7!}{(7-1)!\,1!} = \frac{7!}{6!\times 1!} = 7$$

$$\binom{7}{2} = \frac{7!}{(7-2)!\,2!} = \frac{7!}{5!\times 2!} = \frac{7\times 6\times 5!}{5!\times 2!} = \frac{42}{2} = 21$$

$$\binom{7}{3} = \frac{7!}{(7-3)!\,3!} = \frac{7\times 6\times 5\times 4!}{4!\times 3!} = \frac{210}{6} = 35$$

$$\binom{7}{4}=\frac{7!}{(7-4)!\,4!}=\frac{7\times6\times5\times4!}{3!\times4!}=\frac{210}{3!}=\frac{210}{6}=35$$

$$\binom{7}{5}=\frac{7!}{(7-5)!\,5!}=\frac{7\times6\times5!}{2!\times5!}=\frac{42}{2!}=\frac{42}{2}=21$$

$$(1+x)^7=\binom{7}{0}1^7+\binom{7}{1}1^6x+\binom{7}{2}1^5x^2+\binom{7}{3}1^4x^3+\binom{7}{4}1^3x^4+\binom{7}{5}1^2x^5+\ldots$$

$$(1+x)^7=1+7x+21x^2+35x^3+35x^4+21x^5+\ldots$$

\ $(1+x)^7=1+7x+21x^2+35x^3+35x^4+21x^5+\ldots$

(b)

$$(a+b)^n=\frac{a^n}{0!}+\frac{n\times a^{n-1}b}{1!}+\frac{n(n-1)\times a^{n-2}b^2}{2!}+\frac{n(n-1)(n-2)\times a^{n-3}b^3}{3!}+\ldots+b^n$$

$$(1.01)^7=(1+0.01)^7=\frac{1^7}{0!}+\frac{7\times1^6(0.01)}{1!}+\frac{7(6)\times1^5(0.01)^2}{2!}+\frac{7(6)(5)\times1^4(0.01)^3}{3!}+$$

$$\therefore(1+0.01)^7=1+7(0.01)+21(0.01)^2+35(0.01)^3+35(0.01)^4+\ldots$$

$$\therefore(1+0.01)^7\approx1+0.07+0.0021+0.000035+0.00000035=1.07213535\simeq1.0721354$$

(c) $(1+x)^7=1+7x+21x^2+35x^3+35x^4+21x^5+\ldots$

$$\therefore(1-2x)^7=1+7(-2x)+21(-2x)^2+35(-2x)^3+35(-2x)^4+21(-2x)^5+\ldots$$

$$\therefore(1-2x)^7=1-14x+84x^2-280x^3+560x^4-672x^5+\ldots$$

Example: Express $\left(2-\sqrt{3}\right)^4$ in the form $m+n\sqrt{3}$.

Ans: $\left(2-\sqrt{3}\right)^4=\binom{4}{0}2^4+\binom{4}{1}2^3\left(-\sqrt{3}\right)+\binom{4}{2}2^2\left(-\sqrt{3}\right)^2+\binom{4}{3}2^1\left(-\sqrt{3}\right)^3+\binom{4}{4}2^0\left(-\sqrt{3}\right)^4$

$$\binom{4}{0}=\frac{4!}{(4-0)!\,0!}=\frac{4!}{4!\times0!}=1 \qquad \binom{4}{1}=\frac{4!}{(4-1)!\,1!}=\frac{4\times3!}{3!\times1!}=4$$

$$\binom{4}{2}=\frac{4!}{(4-2)!\,2!}=\frac{4!}{2!\times2!}=\frac{4\times3!}{2!\times2}=3!=3\times2=6$$

$$\binom{4}{3}=\frac{4!}{(4-3)!\,3!}=\frac{4\times3!}{1!\times3!}=4 \qquad \binom{4}{4}=\frac{4!}{(4-4)!\,4!}=\frac{4!}{0!\times4!}=\frac{4!}{4!}=1$$

$$\left(2-\sqrt{3}\right)^4=\binom{4}{0}2^4+\binom{4}{1}2^3\left(-\sqrt{3}\right)+\binom{4}{2}2^2\left(-\sqrt{3}\right)^2+\binom{4}{3}2^1\left(-\sqrt{3}\right)^3+\binom{4}{4}2^0\left(-\sqrt{3}\right)^4$$

$$\left(2-\sqrt{3}\right)^4=2^4+4\times2^3\left(-\sqrt{3}\right)+6\times2^2\left(-\sqrt{3}\right)^2+4\times2^1\left(-\sqrt{3}\right)^3+1\times2^0\left(-\sqrt{3}\right)^4$$

$$\therefore\ \left(2-\sqrt{3}\right)^4=16-32\left(\sqrt{3}\right)+72-24\left(\sqrt{3}\right)+9=97-56\sqrt{3}$$

Example: If the **first 3 terms** in the binomial expansion of $(1+ax)^b$ are: $1-12x+54x^2$. Find the values of ***a*** and ***b***.

Ans: $(1+ax)^b=\binom{b}{0}1^b+\binom{b}{1}1^{b-1}(ax)+\binom{b}{2}1^{b-2}(ax)^2+\ldots$

$$\binom{b}{0}=\frac{b!}{(b-0)!\,0!}=\frac{b!}{b!\times0!}=1 \qquad \binom{b}{1}=\frac{b!}{(b-1)!\,1!}=\frac{b\times(b-1)!}{(b-1)!\,1!}=b$$

$$\binom{b}{2}=\frac{b!}{(b-2)!\,2!}=\frac{b(b-1)(b-2)!}{(b-2)!\times2!}=\frac{b(b-1)}{2!}=\frac{b(b-1)}{2}$$

$$(1+ax)^b=1+b(ax)+\frac{b(b-1)}{2}(ax)^2+\ldots \qquad \textbf{(1)}$$

$$(1+ax)^b=1+abx+\frac{b(b-1)}{2}a^2x^2+\ldots \qquad \textbf{(1)}$$

$$(1+ax)^b=1-12x+54x^2+\ldots \qquad \textbf{(2)}$$

Comparing Eq.1 and Eq.2:

$$ab=-12 \text{ Þ } a=-\frac{12}{b}$$

$$\frac{b(b-1)a^2}{2}=54\Rightarrow\frac{b(b-1)\left(-\frac{12}{b}\right)^2}{2}=\frac{144\left(b^2-b\right)}{2b^2}=54$$

$$\frac{144\left(b^2-b\right)}{2b^2}=54\Rightarrow108b^2=144b^2-144b\Rightarrow-36b^2=-144b\Rightarrow b=4$$

and $a=-\frac{12}{b}=-\frac{12}{4}=-3$

Example: (a) Find the value of $(0.99)^7$ correct to **5** decimal places, without using a calculator.

Ans: **Let x = 0.01 in the following expansion.**

$$(1-x)^7 = \binom{7}{0}1^7 - \binom{7}{1}1^6 x + \binom{7}{2}1^5 x^2 - \binom{7}{3}1^4 x^3 + \binom{7}{4}1^3 x^4 - \binom{7}{5}1^2 x^5 + \ldots$$

$$\binom{7}{0} = \frac{7!}{(7-0)!\,0!} = \frac{7!}{7!\times 0!} = 1 \qquad \binom{7}{1} = \frac{7!}{(7-1)!\,1!} = \frac{7!}{6!\times 1!} = 7$$

$$\binom{7}{2} = \frac{7!}{(7-2)!\,2!} = \frac{7!}{5!\times 2!} = \frac{7\times 6\times 5!}{5!\times 2!} = \frac{42}{2} = 21$$

$$\binom{7}{3} = \frac{7!}{(7-3)!\,3!} = \frac{7\times 6\times 5\times 4!}{4!\times 3!} = \frac{210}{6} = 35$$

$$\binom{7}{4} = \frac{7!}{(7-4)!\,4!} = \frac{7\times 6\times 5\times 4!}{3!\times 4!} = \frac{210}{3!} = \frac{210}{6} = 35$$

$$\binom{7}{5} = \frac{7!}{(7-5)!\,5!} = \frac{7\times 6\times 5!}{2!\times 5!} = \frac{42}{2!} = \frac{42}{2} = 21$$

$$(1-x)^7 = \binom{7}{0}1^7 - \binom{7}{1}1^6 x + \binom{7}{2}1^5 x^2 - \binom{7}{3}1^4 x^3 + \binom{7}{4}1^3 x^4 - \binom{7}{5}1^2 x^5 + \ldots$$

$$(1-x)^7 = 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + \ldots$$

$$(0.99)^7 = (1-0.01)^7 = 1 - 7(0.01) + 21(0.01)^2 - 35(0.01)^3 + 35(0.01)^4 - 21(0.01)^5 + \ldots$$

$$\therefore\ (0.99)^7 = (1-0.01)^7 = 1 - 0.07 + 0.0021 - 0.000035 + 0.00000035 - 0.0000000021 + \ldots \square\ 0.93207$$

$\therefore\ (0.99)^7 = 0.932065347 \simeq 0.93207$ **OK**

Example: If $(2+kx)^6 = 64 - 576x + cx^2 + \ldots$, find the values of ***k*** and ***c***.

Ans:

$$(a+b)^n = \frac{a^n}{0!} + \frac{n\times a^{n-1}b}{1!} + \frac{n(n-1)\times a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)\times a^{n-3}b^3}{3!} + \ldots + b^n(2+kx)^6$$

$$= \frac{2^6}{0!} + \frac{6\times 2^{6-1}(kx)}{1!} + \frac{6(6-1)\times 2^{6-2}(kx)^2}{2!} + \ldots$$

$$(2+kx)^6 = 64 + 192kx + 240k^2x^2 + \ldots = 64 - 576x + cx^2 + \ldots$$

By comparison: $192kx = -576x \Rightarrow 192k = -576 \Rightarrow k = -\frac{576}{192} = -3$

$$240k^2 = c \Rightarrow c = 240k^2 = 240(-3)^2 = 2160$$

Example: Find the **first 4 terms** in the binomial expansion of $\frac{1}{1+x}$.

State the set of values of ***x*** for which the expansion is **valid**.

Ans: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r$

$$\frac{1}{1+x} = [1+x]^{-1} = \left[1 - x + \frac{(-1)(-1-1)}{2!}(x)^2 + \frac{(-1)(-1-1)(-1-1-1)}{3!}(x)^3 \ldots\right]$$

So: $\frac{1}{1+x} = \left[1 - x + x^2 - \frac{6}{3!}x^3 \ldots\right] = 1 - x + x^2 - x^3 + \ldots$, **valid** for $|x| < 1$

Example: Find the **first 4 terms** in the binomial expansion of $\frac{1}{5-x}$.

State the set of values of ***x*** for which the expansion is **valid**.

Ans: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r$

$$\frac{1}{5-x} = [5-x]^{-1} = \left[5\left(1 - \frac{x}{5}\right)\right]^{-1} = 5^{-1}\left(1 - \frac{x}{5}\right)^{-1} = \frac{1}{5}\left(1 - \frac{x}{5}\right)^{-1}$$

$$= \frac{1}{5}\left[1 + (-1)\left(-\frac{x}{5}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{5}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(-\frac{x}{5}\right)^3 + \ldots\right]$$

$$\therefore \frac{1}{5-x} = \frac{1}{5}\left[1 + \frac{x}{5} + \frac{x^2}{25} + \frac{x^3}{125} + \ldots\right] = \frac{1}{5} + \frac{1}{25}x + \frac{1}{125}x^2 + \frac{1}{625}x^3 + \ldots$$

The expansion is valid for $\left|\frac{x}{5}\right| < 1 \Rightarrow |x| < 5$

Example: **(a)** Find the **first 3 terms** in the binomial expansion of $\frac{1}{\sqrt{1-2x}}$.

State the set of values of ***x*** for which the expansion is **valid**.

Ans: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r$

$$\frac{1}{\sqrt{1-2x}} = \frac{1}{(1-2x)^{\frac{1}{2}}} = [1+(-2x)]^{-\frac{1}{2}}$$

$$= \left[1 + \left(-\frac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2}(-2x)^2 + \ldots\right]$$

$\Rightarrow \frac{1}{\sqrt{1-2x}} = \left[1 + x + \frac{3}{2}x^2 + \ldots\right]$, **Valid for** $|-2x| < 1 \Rightarrow -1 < -2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

(b) Hence find the first three terms in the series expansion of $\frac{1+2x}{\sqrt{1-2x}}$.

Ans: $\frac{1+2x}{\sqrt{1-2x}} = \frac{1+2x}{(1-2x)^{\frac{1}{2}}} = (1+2x)(1+(-2x))^{-\frac{1}{2}} = (1+2x)\left[1 + x + \frac{3}{2}x^2 + \ldots\right]$

$$\Rightarrow \frac{1+2x}{\sqrt{1-2x}} = 1 + x + \frac{3}{2}x^2 + 2x + 2x^2 + 3x^3 + \ldots = 1 + 3x + \frac{7}{2}x^2 + 3x^3 + \ldots$$

Example: Find the coefficient of x^3 in the expansion of $(1-2x)^6$.

Ans: $(a-b)^n = \frac{a^n}{0!} - \frac{n \times a^{n-1}b}{1!} + \frac{n(n-1) \times a^{n-2}b^2}{2!} - \frac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + \ldots - b^n$

$$\Rightarrow (1-2x)^6 = \frac{1^6}{0!} + \frac{6 \times 1^5(-2x)}{1!} + \frac{6 \times 5 \times 1^4 \times (-2x)^2}{2!} + \frac{6 \times 5 \times 4 \times 1^3(-2x)^3}{3!} + \ldots$$

The **coefficient** of x^3 is in: $\frac{6 \times 5 \times 4 \times 1^3(-2x)^3}{3!} = -\frac{960}{6} = -160x^3$

Therefore, the **coefficient of** x^3 **is: -160**

OR use the following method, when n is a positive integer!

$$(a-b)^6 = \binom{6}{0}a^6 - \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 - \binom{6}{3}a^3b^3 + \ldots$$

Looking there properly: $(1-2x)^6 = \ldots + \binom{6}{3}1^3(-2x)^3 + \ldots$

$\binom{6}{3} = \frac{6!}{(6-3)!\,3!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \frac{120}{6} = 20$ **and** $1^3(-2x)^3 = -8x^3$

$\Rightarrow \binom{6}{3}1^3(-2x)^3 = -20 \times 8x^3 = -160x^3$, and the **coefficient is: -160**

Example: Calculate the **coefficient** of x^4 in the expansion of $(x+5)^6$.

Ans: $(a+b)^n = \frac{a^n}{0!} + \frac{n \times a^{n-1} b}{1!} + \frac{n(n-1) \times a^{n-2} b^2}{2!} + \frac{n(n-1)(n-2) \times a^{n-3} b^3}{3!} + \ldots + b^n$

$$(x+5)^6 = \frac{x^6}{0!} + \frac{6 \times x^5(5)}{1!} + \frac{6(5) \times x^4(5)^2}{2!} + \frac{6(5)(4) \times x^3(5)^3}{3!} + \ldots + 5^6$$

$$\therefore (x+5)^6 = x^6 + 30\ x^5 + 375\ x^4 + 2500\ \ x^3 + 9375\ \ x^2 + 18750\ x + 15625$$

Therefore, the **coefficient** of x^4 **is 375**.

OR use the following method, when n is a positive integer!

$$(a+b)^6 = \binom{6}{0} a^6 + \binom{6}{1} a^5 b + \binom{6}{2} a^4 b^2 + \binom{6}{3} a^3 b^3 + \binom{6}{4} a^2 b^4 + \ldots$$

Looking there properly: $(x+5)^6 = \ldots + \binom{6}{4} x^4 (5)^2 + \ldots$

$$\binom{6}{4} = \frac{6!}{(6-4)!\,4!} = \frac{6 \times 5 \times 4!}{2! \times 4!} = \frac{30}{2} = 15 \quad \textbf{and} \quad x^4 (5)^2 = 25x^4$$

$\Rightarrow \binom{6}{4} x^4 (5)^2 = 15 \times 25x^4 = 375x^4$, **and** the **coefficient** of x^4 **is 375**.

Example: Find the coefficient of x^3 in the expansion of $(3-2x)^5$.

Ans: $(a-b)^n = \frac{a^n}{0!} + \frac{n \times a^{n-1}(-b)}{1!} + \frac{n(n-1) \times a^{n-2}(-b)^2}{2!} + \frac{n(n-1)(n-2) \times a^{n-3}(-b)^3}{3!} + \ldots - b^n$

$$(3-2x)^5 = \frac{3^5}{0!} + \frac{5 \times 3^4 (-2x)}{1!} + \frac{5 \times 4 \times 3^3 (-2x)^2}{2!} + \frac{5 \times 4 \times 3 \times 3^2 (-2x)^3}{3!} + \ldots$$

The term x^3 is: $\frac{5 \times 4 \times 3 \times 3^2 (-2x)^3}{3!} = -\frac{4320}{6} x^3 = -720x^3$

Therefore, coefficient of x^3 **is: -720**

OR use the following method, when n is a positive integer!

$$(a+b)^6 = \binom{6}{0} a^6 + \binom{6}{1} a^5 b + \binom{6}{2} a^4 b^2 + \binom{6}{3} a^3 b^3 + \binom{6}{4} a^2 b^4 + \ldots$$

Looking there properly: $(3-2x)^5 = \ldots + \binom{5}{3}(3)^2(-2x)^3 + \ldots$

$$\binom{5}{3} = \frac{5!}{(5-3)!\,3!} = \frac{5\times4\times3!}{2!\times3!} = \frac{20}{2} = 10 \quad \textbf{and} \quad 3^2(-2x)^3 = -72x^3$$

$\Rightarrow \binom{5}{3}(3)^2(-2x)^3 = -720x^3$, hence, the coefficient of x^3 **is: -720**

Example: (i) Find the value of ${}^8C_3$.

Ans: $${}^8C_3 = \frac{8!}{(8-3)!\,3!} = \frac{8\times7\times6\times5!}{5!3!} = 56$$

(ii) Find the coefficient of x^3 in the binomial expansion of $\left(1-\frac{1}{2}x\right)^8$.

Ans: $(a-b)^n = \frac{a^n}{0!} - \frac{n\times a^{n-1}b}{1!} + \frac{n(n-1)\times a^{n-2}b^2}{2!} - \frac{n(n-1)(n-2)\times a^{n-3}b^3}{3!} + \ldots - b^n$

$$\left(1-\frac{1}{2}x\right)^8 = \frac{1^8}{0!} + \frac{8\times1^7\times\left(-\frac{1}{2}x\right)}{1!} + \frac{8\times7\times1^6\times\left(-\frac{1}{2}x\right)^2}{2!} + \frac{8\times7\times6\times1^5\times\left(-\frac{1}{2}x\right)^3}{3!} + \ldots$$

$$\frac{8\times7\times6\times1^5\times\left(-\frac{1}{2}x\right)^3}{3!} = \frac{336}{6}\times\left(-\frac{x^3}{8}\right) = -7\,x^3$$

The coefficient of x^3 **is: -7**

OR use the following method, when n is a positive integer!

$$(a+b)^8 = \binom{8}{0}a^8 + \binom{8}{1}a^7b + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 + \ldots$$

Looking there properly: $\left(1-\frac{1}{2}x\right)^8 = \ldots + \binom{8}{3}(1)^5\left(-\frac{1}{2}x\right)^3 + \ldots$

$$\binom{8}{3}=\frac{8!}{(8-3)!\,3!}=\frac{8\times7\times6\times5!}{5!\times3!}=56 \quad \textbf{and} \quad (1)^5\left(-\frac{1}{2}x\right)^3=-\frac{x^3}{8}$$

$$\Rightarrow\binom{8}{3}(1)^5\left(-\frac{1}{2}x\right)^3=56\times\left(-\frac{x^3}{8}\right)=-7x^3$$

The coefficient of x^3 **is: -7**

Example: Find the **coefficient** of x^3 in the binomial expansion of $(5-2x)^5$.

Ans: $(a-b)^n=\frac{a^n}{0!}+\frac{n\times a^{n-1}(-b)}{1!}+\frac{n(n-1)\times a^{n-2}(-b)^2}{2!}-\frac{n(n-1)(n-2)\times a^{n-3}(-b)^3}{3!}+\ldots-b^n$

$$(5-2x)^5=\frac{5^5}{0!}+\frac{5\times5^4(-2x)}{1!}+\frac{5\times4\times5^3(-2x)^2}{2!}+\frac{5\times4\times3\times5^2(-2x)^3}{3!}+\ldots$$

Hence, the term of x^3 is: $\frac{5\times4\times3\times5^2(-2x)^3}{3!}=-\frac{1500}{6}x^3=-2000x^3$

Therefore, the **coefficient** of x^3 **is: -2000**

OR use the following method, when n is a positive integer!

$$(a+b)^5=\binom{5}{0}a^5+\binom{5}{1}a^4b+\binom{5}{2}a^3b^2+\binom{5}{3}a^2b^3+\ldots$$

Looking there properly: $(5-2x)^5=\ldots+\binom{5}{3}(5)^2(-2x)^3+\ldots$

$$\binom{5}{3}=\frac{5!}{(5-3)!\,3!}=\frac{5\times4\times3!}{2!\times3!}=\frac{20}{2}=10 \quad \textbf{and} \quad 5^2(-2x)^3=-200x^3$$

$\Rightarrow\binom{5}{3}(5)^2(-2x)^3=-2000x^3$, hence the **coefficient** of x^3 **is: -2000**

Example: (i) Find the coefficient of x^3 in the expansion of $(x^2-3)(x^3+7x+1)$.

Ans: $(x^2-3)(x^3+7x+1)=\ldots.+7x^3-3x^3+\ldots..$

The coefficient of x^3 is: **4**

(ii) Find the coefficient of x^2 in the binomial expansion of $(1+2x)^7$.

Ans: $(a+b)^n = \frac{a^n}{0!} + \frac{n\times a^{n-1}b}{1!} + \frac{n(n-1)\times a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)\times a^{n-3}b^3}{3!} + \ldots + b^n$

$$(1+2x)^7 = \frac{1^7}{0!} + \frac{7\times 1^6(2x)}{1!} + \frac{7\times 6\times 1^5(2x)^2}{2!} + \ldots$$

$\Rightarrow \frac{7\times 6\times 1^5(2x)^2}{2!} = \frac{168}{2}x^2 = 84x^2$, hence the **coefficient** of x^2 **is: 84**

OR use the following method, when n is a positive integer!

$$(a+b)^7 = \binom{7}{0}a^7 + \binom{7}{1}a^6b + \binom{7}{2}a^5b^2 + \ldots$$

Looking there properly: $(1+2x)^7 = \ldots + \binom{7}{2}(1)^5(2x)^2 + \ldots$

$\binom{7}{2} = \frac{7!}{(7-2)!\,2!} = \frac{7\times 6\times 5!}{5!\times 2!} = \frac{42}{2} = 21$ **and** $1^5(2x)^2 = 4x^2$

$\Rightarrow \binom{7}{2}(1)^5(2x)^2 = 84x^2$, hence the coefficient of x^2 **is: 84**

Example: Find the **coefficient** of x^5 in the binomial expansion of $(5+2x)^7$.

Ans:

$$(a+b)^7 = \binom{7}{0}a^7 + \binom{7}{1}a^6b + \binom{7}{2}a^5b^2 + \binom{7}{3}a^4b^3 + \binom{7}{4}a^3b^4 + \binom{7}{5}a^2b^5 + \binom{7}{6}ab^6 + \binom{7}{7}b^7$$

looking there properly: $(5+2x)^7 = \ldots + \binom{7}{5}5^2(2x)^5 + \ldots$

$\binom{7}{5} = \frac{7!}{(7-5)!\,5!} = \frac{7\times 6\times 5!}{2!\times 5!} = 21$ **and** $5^2(2x)^5 = 25\times 2^5x^5 = 800x^5$

$\Rightarrow \binom{7}{5}5^2(2x)^5 = 21\times 25\times 32x^5 = 16800x^5 \ldots$, **coefficient is: 16800**

Example: Find the **first four terms** in the binomial expansion of $(1+3x)^{\frac{1}{3}}$

State the **range of values of *x*** for which the expansion is valid.

Ans:

General Case:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots$$

$$(1+b)^n = 1 + \frac{nb}{1!} + \frac{n(n-1)b^2}{2!} + \frac{n(n-1)(n-2)b^3}{3!} + \frac{n(n-1)(n-2)(n-3)b^4}{4!} + \ldots$$

$$(1+3x)^{\frac{1}{3}} = 1 + \frac{\left(\frac{1}{3}\right)(3x)}{1!} + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)(3x)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)(3x)^3}{3!} + \ldots$$

$\therefore\ (1+3x)^{\frac{1}{3}} = 1 + x - x^2 + \frac{5}{3}x^3 + \ldots$, valid for $-1 < 3x < 1$ **or** $-\frac{1}{3} < x < \frac{1}{3}$

Note: when ***n*** is real, but **not a positive integer**, then an endless succession of terms is obtained. Such expansions are valid **if** $|b| < |a|$.

Example: Write the binomial expansion of $(a+b)^{-5}$, where $|b| < |a|$.

Ans: $(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots$

$$(a+b)^{-5} = a^{-5} + \frac{(-5)a^{-6}b}{1!} + \frac{(-5)(-6)a^{-7}b^2}{2!} + \frac{(-5)(-6)(-7)a^{-8}b^3}{3!} + \frac{(-5)(-6)(-7)(-8)a^{-9}b^4}{4!} + \frac{(-5)(-6)(-7)(-8)(-9)a^{-10}b^5}{5!}$$

$$(a+b)^{-5} = \frac{1}{a^5} - \frac{5b}{a^6} + \frac{15b^2}{a^7} - \frac{35b^3}{a^8} + \frac{70b^4}{a^9} - \frac{126b^5}{a^{10}}$$

$$(a+b)^{-5} = \frac{1}{(a+b)^5} = \frac{1}{a^5 + \frac{5a^4b}{1!} + \frac{(5)(4)a^3b^2}{2!} + \frac{5(4)(3)a^2b^3}{3!} + \frac{5(4)(3)(2)ab^4}{4!} + \frac{5(4)(3)(2)(1)b^5}{5!} \ldots}$$

$$(a+b)^{-5} = \frac{1}{(a+b)^5} = \frac{1}{a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5}$$

$$\frac{1}{a^5}-\frac{5b}{a^6}+\frac{15b^2}{a^7}-\frac{35b^3}{a^8}+\frac{70b^4}{a^9}-\frac{126b^5}{a^{10}}$$

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \;\big)\; 1$$

$$-\left(1+\frac{5b}{a}+\frac{10b^2}{a^2}+\frac{10b^3}{a^3}+\frac{5b^4}{a^4}+\frac{b^5}{a^5}\right)$$

$$-\frac{5b}{a}-\frac{10b^2}{a^2}-\frac{10b^3}{a^3}-\frac{5b^4}{a^4}-\frac{b^5}{a^5}$$

$$+\left(-\frac{5b}{a}-\frac{25b^2}{a^2}-\frac{50b^3}{a^3}-\frac{50b^4}{a^4}-\frac{25b^5}{a^5}-\frac{5b^6}{a^6}\right)$$

$$\frac{15b^2}{a^2}+\frac{40b^3}{a^3}+\frac{45b^4}{a^4}+\frac{24b^5}{a^5}+\frac{5b^6}{a^6}$$

$$-\left(\frac{15b^2}{a^2}+\frac{75b^3}{a^3}+\frac{150b^4}{a^4}+\frac{150b^5}{a^5}+\frac{75b^6}{a^6}+\frac{15b^7}{a^7}\right)$$

$$-\frac{35b^3}{a^3}-\frac{105b^4}{a^4}-\frac{126b^5}{a^5}-\frac{70b^6}{a^6}-\frac{15b^7}{a^7}$$

$$+\left(-\frac{35b^3}{a^3}-\frac{175b^4}{a^4}-\frac{350b^5}{a^5}-\frac{350b^6}{a^6}-\frac{175b^7}{a^7}-\frac{35b^8}{a^8}\right)$$

$$\frac{70b^4}{a^4}+\frac{224b^5}{a^5}+\frac{280b^6}{a^6}+\frac{160b^7}{a^7}+\frac{35b^8}{a^8}$$

$$-\left(\frac{70b^4}{a^4}+\frac{350b^5}{a^5}+\frac{700b^6}{a^6}+\frac{700b^7}{a^7}+\frac{350b^8}{a^8}+\frac{70b^9}{a^9}\right)$$

$$-\frac{126b^5}{a^5}-\frac{420b^6}{a^6}-\frac{540b^7}{a^7}-\frac{315b^8}{a^8}-\frac{70b^9}{a^9}$$

$$+\left(-\frac{126b^5}{a^5}-\frac{630b^6}{a^6}-\frac{1260b^7}{a^7}-\frac{1260b^8}{a^8}-\frac{630b^9}{a^9}-\frac{126b^{10}}{a^{10}}\right)$$

$$\frac{210b^6}{a^6}+\frac{720b^7}{a^7}+\frac{720b^8}{a^8}+\frac{945b^9}{a^9}+\frac{560b^{10}}{a^{10}}$$

Example: Find the coefficient of x^5 term in the expansion of $(3-2x)^7$.

Ans:

The term involving x^5 is given by: $\binom{7}{5}(3)^2(-2x)^5=\frac{7!}{(7-5)!5!}(3)^2(-2x)^5$

$$\frac{7!}{2!\,5!}(9)(-32x^5)=\frac{7\times8\times7\times6\times5!}{2!\,5!}(9)(-32x^5)=-338688x^5$$

Hence, the coefficient of x^5 is **-338688**

EXERCISES

1. Express $\frac{3x-1}{(x+1)(x-3)}$ in partial fractions.

2. Express $\frac{10x-39}{(x-4)(2x+3)}$ in partial fractions.

3. Express $\frac{2}{x(x^2+3x+2)}$ in partial fractions.

4. Express $\dfrac{1}{(x+1)(x^2+2x+2)}$ in partial fractions.

5. Express $\dfrac{-40x+18}{5x^2+18x}$ in partial fractions.

6. Express $\dfrac{x^2-3x-40}{(x-4)(x+3)}$ in partial fractions.

7. Express $\dfrac{x+2}{x(x^2+2x+1)}$ in partial fractions.

8. Express $\dfrac{x^4+x^3+x^2+x+1}{(x-1)^5}$ in partial fractions.

9. Express $\dfrac{2x^3+x^2-8x+7}{(x-1)(x-2)(x-3)}$ in partial fractions.

10. Find the binomial expansion of $(2+x)^{-3}$, writing each term as simply as possible.

11. Expand $\sqrt{\dfrac{1+x}{1-2x}}$ as a series of ascending powers of x up to and including the term containing x^2.

12. Find **a quadratic approximation for** $(3+5x)^{-2}$, giving your answer in the form $a+bx+cx^2$. State the range of values of **x** for which the expansion is valid.

13. Assuming x is too small that x^3 and higher power of x can be neglected, show that:

(a) $\frac{1}{\sqrt{1+x}} \cong 1-\frac{x}{2}+\frac{3x^2}{8}$

(b) By letting $x=\dfrac{1}{4}$, find rational approximation for $\sqrt{5}$.

(c) Hence find the first three terms in the series expansion of $\dfrac{1+2x}{\sqrt{1+x}}$.

14. Find the expression of $\frac{(1+x)^3}{2-x}$ up to and including the term in x^3. Hence evaluate $(1.2)^3$ giving your answer to two decimal places.

15. Express $\frac{2x+5}{(x+1)(x+2)}$ in partial fractions and hence find the first four terms in the series expansion of $\frac{2x+5}{(x+1)(x+2)}$. State the value of **x** for which the expansion is valid.

16. Express $\frac{3x-1}{(x+1)(x-3)}$ in partial fractions and hence find the first four terms in the series expansion of $\frac{3x-1}{(x+1)(x-3)}$. State the value of **x** for which the expansion is valid.

17. Find the series expansion of $\left(\frac{1}{x+1}\right)\left(\frac{2}{x-3}\right)$.

18. Find the expression of $\frac{(1-x)^3}{3-x}$ up to and including the term in x^3. Hence evaluate $(0.97)^3$ giving your answer to five decimal places.

19. If $(1+ax)^k = 1+24x+264x^2+\ldots$, find the values of ***a*** and ***k***.

EXERCISES and SOLUTIONS

1. Express $\frac{3x-1}{(x+1)(x-3)}$ in partial fractions.

Ans: $\frac{3x-1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$, **where** $A = \left.\frac{3x-1}{x-3}\right|_{x=-1} = 1$

and $B = \left.\frac{3x-1}{x+1}\right|_{x=3} = 2$ \ $\frac{3x-1}{(x+1)(x-3)} = \frac{1}{x+1} + \frac{2}{x-3}$

2. Express $\frac{10x-39}{(x-4)(2x+3)}$ in partial fractions.

Ans: $\frac{10x-39}{(x-4)(2x+3)} = \frac{A}{x-4} + \frac{B}{2x+3}$, **where**

$$A = \left.\frac{10x - 39}{2x + 3}\right|_{x = 4} = \frac{1}{11} \quad \textbf{and} \quad B = \left.\frac{10x - 39}{x - 4}\right|_{x = -1.5} = \frac{-44}{-5.5} = 8$$

$$\therefore \frac{10x - 39}{(x-4)(2x+3)} = \frac{1}{11(x-4)} + \frac{8}{(2x+3)}$$

3. Express $\dfrac{2}{x(x^2 + 3x + 2)}$ in partial fractions.

Ans: $\dfrac{2}{x(x+2)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x+1}$, **where**

$$A = \left.\frac{2}{(x+2)(x+1)}\right|_{x = 0} = 1 \quad \textbf{and} \quad B = \left.\frac{2}{x(x+1)}\right|_{x = -2} = \frac{2}{-2(-2+1)} = 1$$

$$C = \left.\frac{2}{x(x+2)}\right|_{x = -1} = \frac{2}{-1(-1+2)} = \frac{2}{-1} = -2$$

$$\backslash \quad \frac{2}{x(x^2 + 3x + 2)} = \frac{2}{x(x+2)(x+1)} = \frac{1}{x} + \frac{1}{x+2} - \frac{2}{x+1}$$

4. Express $\dfrac{1}{(x+1)(x^2 + 2x + 2)}$ in partial fractions.

Ans: $\dfrac{1}{(x+1)(x^2 + 2x + 2)} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2 + 2x + 2}$, **where**

$$A = \left.\frac{1}{x^2 + 2x + 2}\right|_{x = -1} = 1$$

$$\backslash \quad \frac{1}{(x+1)(x^2 + 2x + 2)} = \frac{1}{x+1} + \frac{Bx + C}{x^2 + 2x + 2} = \frac{x^2 + 2x + 2 + (x+1)(Bx + C)}{(x+1)(x^2 + 2x + 2)}$$

$$\Rightarrow x^2 + 2x + 2 + Bx^2 + Cx + Bx + C = 1 \qquad \Rightarrow 2 + C = 1 \qquad \Rightarrow C = -1$$

$$and \quad 2 + C + B = 0 \qquad \Rightarrow B = -1$$

$$\therefore \frac{1}{(x+1)(x^2+2x+2)} = \frac{1}{x+1} - \frac{x+1}{x^2+2x+2} = \frac{1}{x+1} - \frac{x+1}{(x+1)^2+1}$$

5. Express $\dfrac{-40x + 18}{5x^2 + 18x}$ in partial fractions.

Ans: $\frac{-40x+18}{5x\left(x+\frac{18}{5}\right)} = \frac{A}{x} + \frac{B}{\left(x+\frac{18}{5}\right)}$, **where** $\quad A = \left.\frac{-40x+18}{5\left(x+\frac{18}{5}\right)}\right|_{x = 0} = \frac{18}{18} = 1$

and $\quad B = \left.\frac{-40x+18}{5x}\right|_{x = -\frac{18}{5}} = \frac{-40\left(-\frac{18}{5}\right)+18}{5\left(-\frac{18}{5}\right)} = \frac{144+18}{-18} = -\frac{162}{18} = -9$

$$\therefore \frac{-40x+18}{5x^2+18x} = \frac{-40x+18}{5x\left(x+\frac{18}{5}\right)} = \frac{A}{x} + \frac{B}{\left(x+\frac{18}{5}\right)} = \frac{1}{x} - \frac{9}{\left(x+\frac{18}{5}\right)}$$

6. Express $\frac{x^2-3x-40}{(x-4)(x+3)}$ in partial fractions.

Ans:

$$\frac{x^2-3x-40}{(x-4)(x+3)} = \frac{x^2-3x-40}{x^2-x-12} = 1+\frac{-2x-28}{x^2-x-12} = 1-\frac{2x+28}{x^2-x-12} = 1+\frac{2x+28}{(x-4)(x+3)}$$

$\therefore \frac{2x+28}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$, where $A = \left.\frac{2x+28}{x+3}\right|_{x=4} = \frac{8+28}{4+3} = \frac{36}{7}$

and $B = \left.\frac{2x+28}{x-4}\right|_{x=3} = \frac{6+28}{3-4} = \frac{34}{-1} = -34$

$$\therefore \frac{x^2-3x-40}{(x-4)(x+3)} = 1 - \frac{36}{7(x-4)} + \frac{34}{(x+3)}$$

7. Express $\frac{x+2}{x(x^2+2x+1)}$ in partial fractions.

Ans: $\frac{x+2}{x(x^2+2x+1)} = \frac{x+2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, **where**

$A = \left.\frac{x+2}{(x^2+2x+1)}\right|_{x=0} = \frac{2}{1} = 2,$ **and** $C = \left.\frac{x+2}{x}\right|_{x=-1} = \frac{-1+2}{-1} = -1,$

$$B = \left.\frac{d}{dx}\left(\frac{x+2}{x}\right)\right|_{x=-1} = \left.\frac{x-(x+2)}{x^2}\right|_{x=-1} = \left.\frac{-2}{x^2}\right|_{x=-1} = \frac{-2}{(-1)^2} = -2$$

Check: $\frac{x+2}{x(x^2+2x+1)} \stackrel{?}{=} \frac{x+2}{x(x+1)^2} = \frac{2}{x} - \frac{2}{x+1} - \frac{1}{(x+1)^2}$

$$\frac{2}{x} - \frac{2}{x+1} - \frac{1}{(x+1)^2} = \frac{2(x+1)^2 - 2x(x+1) - x}{x(x+1)^2} = \frac{2(x^2+2x+1) - 2x^2 - 2x - x}{x(x+1)^2} = \frac{x+2}{x(x+1)^2}$$

8. Express $\frac{x^4+x^3+x^2+x+1}{(x-1)^5}$ in partial fractions.

Ans: $\frac{x^4+x^3+x^2+x+1}{(x-1)^5} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} + \frac{E}{(x-1)^5}$

Method 1:

$$\frac{x^4+x^3+x^2+x+1}{(x-1)^5}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}+\frac{E}{(x-1)^5}$$

$$\therefore \frac{x^4+x^3+x^2+x+1}{(x-1)^5}=\frac{A(x-1)^4+B(x-1)^3+C(x-1)^2+D(x-1)+E}{(x-1)^5}$$

So: $x^4+x^3+x^2+x+1=A(x-1)^4+B(x-1)^3+C(x-1)^2+D(x-1)+E$

$$=A(x^4-4x^3+6x^2-4x+1)+B(x^3-3x^2+3x-1)+C(x^2-2x+1)+D(x-1)+E$$

By comparison: A = 1 (coefficient of x^4) **and**

$$x^4-4x^3+6x^2-4x+1+Bx^3-3Bx^2+3Bx-B+Cx^2-2Cx+C+Dx-D+E=x^4+x^3+x^2+x+1$$

and $B-4=1 \Rightarrow B=5$ (coefficient of x^3)

and $6-3B+C=1 \Rightarrow C=1-6+3B=-5+3(5)=10$ (coefficient of x^2)

and $-4+3B-2C+D=1 \Rightarrow D=1+4-3(5)+2(10)=10$ (coefficient of x)

and $1-B+C-D+E=1 \Rightarrow E=1-1+B-C+D=5-10+10=5$

So: $$\frac{x^4+x^3+x^2+x+1}{(x-1)^5}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}+\frac{E}{(x-1)^5}$$

$$\therefore \frac{x^4+x^3+x^2+x+1}{(x-1)^5}=\frac{1}{(x-1)}+\frac{5}{(x-1)^2}+\frac{10}{(x-1)^3}+\frac{10}{(x-1)^4}+\frac{5}{(x-1)^5}$$

Method 2:

$$\frac{x^4+x^3+x^2+x+1}{(x-1)^5}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}+\frac{E}{(x-1)^5}$$

To find **E**, remove the denominator of **E** from the original fraction, and then evaluate the remaining fraction at the repeated root of denominator **E**, i.e., in this case at **x = 1**

Therefore, $E=x^4+x^3+x^2+x+1\Big|_{x=1}=1+1+1+1+1=5$

To find **D**, remove the denominator of **E** from the original fraction, and then take the **1st derivative** of the remaining fraction and evaluate that at the repeated root of the denominator of **D**, i.e. at **x = 1**

and $D=\frac{1}{1!}\frac{d}{dx}(x^4+x^3+x^2+x+1)\Big|_{x=1}=(4x^3+3x^2+2x+1)\Big|_{x=1}=4+3+2+1=10$

To find **C**, remove the denominator of **E** from the original fraction, and then take the **2nd derivative** of the remaining fraction and evaluate at the root of denominator **C**, i.e. at **x = 1**

and $C=\frac{1}{2!}\times\frac{d^2}{dx^2}(x^4+x^3+x^2+x+1)\Big|_{x=1}=\frac{1}{2!}(12x^2+6x+2)\Big|_{x=1}=\frac{20}{2}=10$

To find **B**, remove the denominator of **E** from the original fraction, and then take the **3rd derivative** of the remaining fraction and evaluate at the root of denominator **B**, i.e. at **x = 1**

and $B = \frac{1}{3!} \cdot \frac{d^3}{dx^3}(x^4 + x^3 + x^2 + x + 1)\Big|_{x=1} = \frac{1}{3!}(24x + 6)\Big|_{x=1} = \frac{30}{6} = 5$

To find **A**, remove the denominator of **E** from the original fraction, and then take the **4th derivative** of the remaining fraction and evaluate at the root of denominator **A**, i.e. at **x = 1**

and $B = \frac{1}{4!} \times \frac{d^4}{dx^4}(x^4 + x^3 + x^2 + x + 1)\Big|_{x=1} = \frac{1}{4!}(24x + 6)\Big|_{x=1} = \frac{24}{4!} = \frac{24}{24} = 1$

So: $$\frac{x^4 + x^3 + x^2 + x + 1}{(x-1)^5} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} + \frac{E}{(x-1)^5}$$

$$\therefore \frac{x^4+x^3+x^2+x+1}{(x-1)^5} = \frac{1}{(x-1)} + \frac{5}{(x-1)^2} + \frac{10}{(x-1)^3} + \frac{10}{(x-1)^4} + \frac{5}{(x-1)^5}$$

9. Express $\frac{2x^3 + x^2 - 8x + 7}{(x-1)(x-2)(x-3)}$ in partial fractions.

Ans: $$\frac{2x^3 + x^2 - 8x + 7}{(x-1)(x-2)(x-3)} = \frac{2x^3 + x^2 - 8x + 7}{x^3 - 6x^2 + 11x - 6} = A + \frac{B}{(x-1)} + \frac{C}{(x-2)} + \frac{D}{(x-3)}$$

Method 1:

$$\frac{2x^3 + x^2 - 8x + 7}{x^3 - 6x^2 + 11x - 6} = A + \frac{B}{(x-1)} + \frac{C}{(x-2)} + \frac{D}{(x-3)}$$

$$= \frac{A(x-1)(x-2)(x-3) + B(x-2)(x-3) + C(x-1)(x-3) + D(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

So: $2x^3 + x^2 - 8x + 7 = A(x-1)(x-2)(x-3) + B(x-2)(x-3) + C(x-1)(x-3) + D(x-1)(x-2)$

By comparison: A = 2, because of coefficient of x^3

at x = 2, $2(2)^3 + (2)^2 - 8(2) + 7 = C(2-1)(2-3)$

$16 + 4 - 16 + 7 = -C = 11 \Rightarrow C = -11$

at x = 3, $2(3)^3 + (3)^2 - 8(3) + 7 = D(3-1)(3-2)$

$54 + 9 - 24 + 7 = 2D = 46 \Rightarrow D = 23$

at x = 1, $2(1)^3 + (1)^2 - 8(1) + 7 = B(1-2)(1-3)$

$2 + 1 - 8 + 7 = 2B = 2 \Rightarrow B = \frac{2}{2} = 1$

So: $$\frac{2x^3 + x^2 - 8x + 7}{x^3 - 6x^2 + 11x - 6} = 2 + \frac{1}{(x-1)} - \frac{11}{(x-2)} + \frac{23}{(x-3)}$$

Method 2:

Note: Since both numerator and denominator of the original fraction is of the same order, then the original fraction is divided, where the quotient is **2** and the remainder is: $13x^2 - 30x + 19$.

Long Division

$$
\begin{array}{r|l}
 & \quad 2 \\
\hline
x^3 - 6x^2 + 11x - 6 & 2x^3 + x^2 - 8x + 7 \\
 & \underline{-2x^3 + 12x^2 - 22x + 12} \\
 & \qquad 13x^2 - 30x + 19 \rightarrow \text{remainder}
\end{array}
$$

$$\frac{2x^3 + x^2 - 8x + 7}{x^3 - 6x^2 + 11x - 6} = 2 + \frac{13x^2 - 30x + 19}{(x-1)(x-2)(x-3)} = 2 + \frac{B}{(x-1)} + \frac{C}{(x-2)} + \frac{D}{(x-3)}$$

$$\Rightarrow \frac{13x^2 - 30x + 19}{(x-1)(x-2)(x-3)} = \frac{B}{(x-1)} + \frac{C}{(x-2)} + \frac{D}{(x-3)} \qquad \text{...(1)}$$

To find **D** in **Eq.1**, remove the denominator of **D** from $\frac{13x^2 - 30x + 19}{(x-1)(x-2)(x-3)}$, and evaluate the remaining fraction $\frac{13x^2 - 30x + 19}{(x-1)(x-2)}$ at the root of denominator **D**, i.e. at **x = 3**

Therefore,

$$D = \left.\frac{13x^2 - 30x + 19}{(x-1)(x-2)}\right|_{x=3} = \frac{13(3)^2 - 30(3) + 19}{(3-1)(3-2)} = \frac{117 - 90 + 19}{2} = \frac{46}{2} = 23$$

To find **C** in **Eq.1**, remove the denominator of **C** from $\frac{13x^2 - 30x + 19}{(x-1)(x-2)(x-3)}$, and evaluate the remaining fraction $\frac{13x^2 - 30x + 19}{(x-1)(x-3)}$ at the roots of the denominator of **C**, i.e. at **x = 2**

Therefore,

$$C = \left.\frac{13x^2 - 30x + 19}{(x-1)(x-3)}\right|_{x=2} = \frac{13(2)^2 - 30(2) + 19}{(2-1)(2-3)} = \frac{52 - 60 + 19}{-1} = -11$$

To find **B** in **Eq.1**, remove the denominator of **B** from $\frac{13x^2 - 30x + 19}{(x-1)(x-2)(x-3)}$, and evaluate the remaining fraction $\frac{13x^2 - 30x + 19}{(x-2)(x-3)}$ at the root of denominator **B**, i.e. at **x = 1**

Therefore,

$$B = \left.\frac{13x^2 - 30x + 19}{(x-2)(x-3)}\right|_{x=1} = \frac{13(1)^2 - 30(1) + 19}{(1-2)(1-3)} = \frac{13 - 30 + 19}{(-1)(-2)} = \frac{2}{2} = 1$$

So:

$$\frac{2x^3 + x^2 - 8x + 7}{x^3 - 6x^2 + 11x - 6} = 2 + \frac{13x^2 - 30x + 19}{(x-1)(x-2)(x-3)} = 2 + \frac{B}{(x-1)} + \frac{C}{(x-2)} + \frac{D}{(x-3)}$$

$$\therefore \frac{2x^3 + x^2 - 8x + 7}{x^3 - 6x^2 + 11x - 6} = 2 + \frac{13x^2 - 30x + 19}{(x-1)(x-2)(x-3)} = 2 + \frac{1}{(x-1)} - \frac{11}{(x-2)} + \frac{23}{(x-3)}$$

10. Find the binomial expansion of $(2+x)^{-3}$, writing each term as simply as possible.

Ans: $(2+x)^{-3} = \left[2\left(1+\frac{x}{2}\right)\right]^{-3} = 2^{-3}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$

$$= \frac{1}{8}\left(\frac{1^{-3}}{0!} + \frac{-3 \times 1^{-3-1}\left(\frac{x}{2}\right)}{1!} + \frac{(-3)(-3-1) \times 1^{-3-2}\left(\frac{x}{2}\right)^2}{2!} + \frac{(-3)(-3-1)(-3-2) \times 1^{-3-3}\left(\frac{x}{2}\right)^3}{3!} + \ldots\right)$$

$$\therefore \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\left(1 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{5}{4}x^3 + \ldots\right)$$

11. Expand $\sqrt{\frac{1+x}{1-2x}}$ as a series of ascending powers of x up to and including the term containing x^2.

Ans: $(1+x)^n = 1^n + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

$$\sqrt{\frac{1+x}{1-2x}} = \frac{(1+x)^{\frac{1}{2}}}{(1-2x)^{\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}} = ?$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\times 1}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3\times 2\times 1}x^3 + \ldots$$

Because the term x^2 is needed:

$$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\times 1}x^2+\ldots = (1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2+\ldots$$

$$\backslash\ (1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$$

Also:

Because the term x^2 is needed:

$$(1-2x)^{-\frac{1}{2}} = 1-\frac{1}{2}(-2x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2\times 1}(-2x)^2+\ldots$$

$$(1-2x)^{-\frac{1}{2}} = 1+x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2\times 1}\left(4x^2\right)+\ldots$$

$$\backslash\ (1-2x)^{-\frac{1}{2}} = 1+x+\frac{3}{2}x^2+\ldots$$

Therefore, $(1+x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}} = \left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)\left(1+x+\frac{3}{2}x^2\right)+\ldots$

$$(1+x)^{\frac{1}{2}}\ (1-2x)^{-\frac{1}{2}} = 1+x+\frac{3}{2}x^2+\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{8}x^2\ldots = 1+\frac{3}{2}x+\left(2-\frac{1}{8}\right)x^2\ldots$$

$$\backslash\ (1+x)^{\frac{1}{2}}\ (1-2x)^{-\frac{1}{2}} = 1+\frac{3}{2}x+\frac{15}{8}x^2\ldots$$

Therefore, the coefficient of x^2 is $\frac{15}{8}$.

12. Find the **a quadratic approximation for** $(3+5x)^{-2}$, giving your answer in the form $a+bx+cx^2$. State the range of values of **x** for which the expansion is valid.

Ans: $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3\ldots\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r$

$$(3+5x)^{-2} = \left[3\left(1+\frac{5x}{3}\right)\right]^{-2} = 3^{-2}\left(1+\frac{5x}{3}\right)^{-2} = \frac{1}{9}\left(1+\frac{5x}{3}\right)^{-2}$$

$$= \frac{1}{9}\left[1-2\left(\frac{5x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{5x}{3}\right)^2+\ldots\right]$$

$$= \frac{1}{9}\left[1-\frac{10}{3}x+\frac{25}{3}x^2+\ldots\right]$$

$$= \frac{1}{9}-\frac{10}{27}x+\frac{25}{27}x^2+\ldots$$

The expansion is **valid** for $\left|\frac{5x}{3}\right| < 1 \Rightarrow |x| < \frac{3}{5}$

So: $(3+5x)^{-2} \approx \frac{1}{9} - \frac{10}{27}x + \frac{25}{27}x^2 + \ldots$

13. Assuming x is too small that x^3 and higher power of x can be neglected, show that:

(a) $\frac{1}{\sqrt{1+x}} \cong 1 - \frac{x}{2} + \frac{3x^2}{8}$

Ans: $\frac{1}{\sqrt{1+x}} = \frac{1}{(1+x)^{\frac{1}{2}}} = [1+x]^{-\frac{1}{2}} = \left[1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2 + \ldots\right]$

\ $\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 + \ldots,$ **Valid for** $|x| < 1$.

(b)

By letting $x = \frac{1}{4}$, find rational approximation for $\sqrt{5}$.

$$\frac{1}{\sqrt{1+\frac{1}{4}}} = 1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{3}{8}\left(\frac{1}{4}\right)^2$$

$$\sqrt{\frac{4}{5}} = 1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{3}{8}\left(\frac{1}{4}\right)^2 = 1 - \frac{1}{8} + \frac{3}{128} = \frac{128-16+3}{128} = \frac{115}{128}$$

$$\sqrt{\frac{4}{5}} = \frac{115}{128} \text{ Þ } \frac{\sqrt{5}}{2} = \frac{128}{115} \text{ Þ } \sqrt{5} = \frac{2' \ 128}{115} = \frac{256}{115}$$

(c) Hence find the first three terms in the series expansion of $\frac{1+2x}{\sqrt{1+x}}$.

Ans: $\frac{1+2x}{\sqrt{1+x}} = \frac{1+2x}{(1+x)^{\frac{1}{2}}} = (1+2x)(1+x)^{-\frac{1}{2}} = (1+2x)\left[1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2 + \ldots\right]$

$$\Rightarrow \frac{1+2x}{\sqrt{1+x}} = (1+2x)\left[1 - \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right]$$

$$\text{Þ } \frac{1+2x}{\sqrt{1-2x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + 2x - x^2 + \frac{3}{4}x^3 + \ldots = 1 + \frac{3}{2}x - \frac{5}{8}x^2 + \frac{3}{4}x^3 + \ldots$$

14. Find the expression of $\frac{(1+x)^3}{2-x}$ up to and including the term in x^3. Hence evaluate $(1.2)^3$ giving your answer to two decimal places.

Ans:

$$\frac{(1+x)^3}{2-x} = (1+x)^3(2-x)^{-1}$$

Expand $(1+x)^3 = 1 + \binom{3}{1}x + \binom{3}{2}x^2 + x^3 = 1 + 3x + 3x^2 + x^3$, valid for all x.

$$(2-x)^{-1} = \left[2\left(1-\frac{x}{2}\right)\right]^{-1} = 2^{-1}\left[1+\left(-\frac{x}{2}\right)\right]^{-1}$$

$$(2-x)^{-1} = \left[2\left(1-\frac{x}{2}\right)\right]^{-1} = 2^{-1}\left[1+\left(-\frac{x}{2}\right)\right]^{-1} = \frac{1}{2}\left[1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2\right]$$

$$= \frac{1}{2}\left[1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots\right] = \frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}+\ldots,$$ Valid for $\left|\frac{x}{2}\right| < 1$ Þ $|x| < 2$

Therefore, $(1+x)^3(2-x)^{-1} = (1+3x+3x^2+x^3)\left(\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}+\ldots\right)$

$$= \frac{1}{2}+\frac{7x}{4}+\frac{19x^2}{8}+\frac{27x^3}{16}+\ldots$$

$$\backslash\ \frac{(1+x)^3}{(2-x)} \text{»} \frac{1}{2}+\frac{7x}{4}+\frac{19x^2}{8}+\frac{27x^3}{16}$$, Valid for $|x| < 2$

Let $x = 0.2$ (which lies in the valid range), then

$$\frac{(1+0.2)^3}{(2-0.2)} \text{»} \frac{1}{2}+\frac{7(0.2)}{4}+\frac{19(0.2)^2}{8}+\frac{27(0.2)^3}{16}$$

$\therefore \frac{(1.2)^3}{(1.8)} \approx 0.9585 \Rightarrow (1.2)^3 = 1.8 \times 0.9585 = 1.73$ (2 d.p)

15. Express $\frac{2x+5}{(x+1)(x+2)}$ in partial fractions and hence find the first four terms in the series expansion of $\frac{2x+5}{(x+1)(x+2)}$. State the value of x for which the expansion is valid.

Ans: $\frac{2x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

Method 1:

$$\frac{2x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

So: $2x+5 = A(x+2)+B(x+1)$

At $x=-1$, $2(-1)+5 = A(-1+2)+B(-1+1) \Rightarrow A = -2+5 = 3$

At $x=-2$, $2(-2)+5 = A(-2+2)+B(-2+1) \Rightarrow -B = -4+5 = 1 \Rightarrow B = -1$

So: $\frac{2x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{3}{x+1} - \frac{1}{x+2}$

Method 2:

$$\frac{2x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

To find A, remove the denominator of A from the original fraction, and evaluate the remaining fraction at the roots of the denominator of A, i.e. at x = -1

Therefore, $A = \left.\frac{2x+5}{x+2}\right|_{x=-1} = \frac{-2+5}{-1+2} = \frac{3}{1} = 3$

To find B, remove the denominator of B from the original fraction, and evaluate the remaining fraction at the roots of the denominator of B, i.e. at x = -2

and $B = \left.\frac{2x+5}{x+1}\right|_{x=-2} = \frac{-4+5}{-2+1} = \frac{1}{-1} = -1$

So: $\frac{2x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{3}{x+1} - \frac{1}{x+2}$

To obtain the series expansion, you need the series expansion of both $\frac{3}{x+1}$ and $\frac{1}{x+2}$.

$$\frac{3}{x+1} = 3(1+x)^{-1} = 3\left[1 + (-1)x + \frac{(-1)(-1-1)}{2!}x^2 + \frac{(-1)(-1-1)(-1-2)}{3!}x^3 + \ldots\right]$$

$= 3 - 3x + 3x^2 - 3x^3 + \ldots$, Valid for $|x| < 1$

$$\frac{1}{x+2} = (x+2)^{-1} = (2+x)^{-1} = \left[2\left(1+\frac{x}{2}\right)\right]^{-1} = 2^{-1}\left[1+\frac{x}{2}\right]^{-1}$$

$$= \frac{1}{2}\left[1+\frac{x}{2}\right]^{-1} = \frac{1}{2}\left[1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{x}{2}\right)^3\right]$$

$$= \frac{1}{2}\left[1+\frac{x}{2}\right]^{-1} = \frac{1}{2}\left[1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right] = \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots$$

Valid for $\left|\frac{x}{2}\right| < 1$ Þ $|x| < 2$

Therefore, $\frac{3}{x+1} - \frac{1}{x+2} = (3 - 3x + 3x^2 - 3x^3 + \ldots) - \left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots\right)$

$= \frac{5}{2} - \frac{11x}{4} + \frac{23x^2}{8} - \frac{47x^3}{16} + \ldots$, Valid for $|x| < 1$, which satisfies both (1) and (2).

16. Express $\frac{3x-1}{(x+1)(x-3)}$ in partial fractions and hence find the first four terms in the series expansion of $\frac{3x-1}{(x+1)(x-3)}$. State the value of x for which the expansion is valid.

Ans: $\frac{3x-1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

Method 1:

$$\frac{3x-1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

So: $3x - 1 = A(x-3) + B(x+1)$

at $x = -1$, $3(-1) - 1 = A(-1-3) + B(-1+1) = -4A = -4 \Rightarrow A = 1$

at $x = 3$, $3(3) - 1 = A(3-3) + B(3+1) = 4B = 9 - 1 = 8 \Rightarrow B = 2$

So: $\frac{3x-1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{1}{x+1} + \frac{2}{x-3}$

Method 2:

$$\frac{3x-1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

To find **A**, remove the denominator of **A** from the original fraction, and evaluate the remaining fraction at the root of denominator **A**, **i.e. at x = -1**

Therefore, $A = \left.\dfrac{3x-1}{x-3}\right|_{x=-1} = \dfrac{-3-1}{-1-3} = \dfrac{-4}{-4} = 1$

To find **B**, remove the denominator of **B** from the original fraction, and evaluate the remaining fraction at the root of denominator **B**, **i.e. at x = 3**

and $B = \left.\dfrac{3x-1}{x+1}\right|_{x=3} = \dfrac{9-1}{3+1} = \dfrac{8}{4} = 2$

So: $\dfrac{3x-1}{(x+1)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x-3} = \dfrac{1}{x+1} + \dfrac{2}{x-3}$

17. Find the series expansion of $\left(\frac{1}{x+1}\right)\left(\frac{2}{x-3}\right)$.

Ans: To obtain the series expansion of $\left(\frac{1}{x+1}\right)\left(\frac{2}{x-3}\right)$, the series expansion of both $\dfrac{1}{x+1}$ **and** $\dfrac{2}{x-3}$ are required.

$$\frac{1}{x+1} = (1+x)^{-1} = \left[1 + (-1)x + \frac{(-1)(-1-1)}{2!}x^2 + \frac{(-1)(-1-1)(-1-2)}{3!}x^3 + \ldots\right]$$

$= 1 - x + x^2 - x^3 + \ldots$, **Valid** for $|x| < 1$

$$\frac{2}{x-3} = 2(x-3)^{-1} = 2(-3+x)^{-1} = 2\left[(-3)\left(1-\frac{x}{3}\right)\right]^{-1} = 2(-3^{-1})\left[1-\frac{x}{3}\right]^{-1}$$

$$= -\frac{2}{3}\left[1-\frac{x}{3}\right]^{-1} = -\frac{2}{3}\left[1 + (-1)\left(-\frac{x}{3}\right) + \frac{(-1)(-1-1)}{2!}\left(-\frac{x}{3}\right)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{3}\right)^3\right]$$

$$= -\frac{2}{3}\left[1-\frac{x}{3}\right]^{-1} = -\frac{2}{3}\left[1 + (-1)\left(-\frac{x}{3}\right) + \frac{(-1)(-1-1)}{2!}\left(-\frac{x}{3}\right)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{3}\right)^3\right]$$

$$= -\frac{2}{3}\left[1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right] = -\frac{2}{3} - \frac{2x}{9} - \frac{2x^2}{27} - \frac{2x^3}{81} + \ldots$$

Valid for $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

18. Find the expression of $\dfrac{(1-x)^3}{3-x}$ up to and including the term in x^3. Hence evaluate $(0.97)^3$ giving your answer to five decimal places.

Ans:

Note: **Because up to x^3** is required, then both $(1-x)^3$ and $\frac{1}{3-x}$ should be expanded maximally up to x^2 as follows:

$$\frac{(1-x)^3}{3-x} = (1-x)^3(3-x)^{-1}$$

$(1-x)^3 = 1 + \binom{3}{1}(-x) + \binom{3}{2}(-x)^2 + \binom{3}{3}(-x)^3 = 1 - 3x + 3x^2 - x^3$, valid for all x.

$$(3-x)^{-1} = \left[3\left(1-\frac{x}{3}\right)\right]^{-1} = 3^{-1}\left[1+\left(-\frac{x}{3}\right)\right]^{-1}$$

$$(3-x)^{-1} = \left[3\left(1-\frac{x}{3}\right)\right]^{-1} = 3^{-1}\left[1+\left(-\frac{x}{3}\right)\right]^{-1} = \frac{1}{3}\left[1^{-1} + (-1)\times 1^{-2}\left(-\frac{x}{3}\right) + \frac{(-2)(-1)}{2!}\times 1^{-3}\left(-\frac{x}{3}\right)^2 + \ldots\right]$$

$= \frac{1}{3}\left[1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right] = \frac{1}{3} + \frac{x}{9} + \frac{x^2}{27} + \ldots$, Valid for $\left|\frac{x}{2}\right| < 1$

i.e. $|x| < 2$

Therefore, $(1-x)^3(3-x)^{-1} = (1-3x+3x^2-x^3)\left(\frac{1}{3} + \frac{x}{9} + \frac{x^2}{27} + \ldots\right)$

$$= \frac{1}{3} + \frac{x}{9} + \frac{x^2}{27} - x - \frac{3x^2}{9} - \frac{3x^3}{27} + x^2 + \frac{3}{9}x^3 - \frac{x^3}{3} + \ldots$$

$\backslash\ \frac{(1-x)^3}{(3-x)} \gg \frac{1}{3} - \frac{8x}{9} + \frac{19x^2}{27} - \frac{x^3}{9} + \ldots$, Valid for $|x| < 2$

Let $x = 0.03$ (which lies in the valid range), then

$$\frac{(1-0.03)^3}{(3-0.03)} \gg \frac{1}{3} - \frac{8(0.03)}{9} + \frac{19(0.03)^2}{27} - \frac{(0.03)^3}{9}$$

$$\frac{(1-0.03)^3}{(3-0.03)} \gg 0.333 - 0.02667 + 0.0006333 - 0.000003$$

$\frac{(1-0.03)^3}{(3-0.03)} \approx 0.3073 \Rightarrow (1-0.03)^3 = (0.97)^3 \approx (0.3073)(2.97) = 0.91268$

$\therefore (0.97)^3 = 0.91267$ (5 d.p)

19. If $(1+ax)^k = 1 + 24x + 264x^2 + \ldots$, find the values of ***a*** and ***k***.

Ans:

$$(1+ax)^k = \binom{k}{0}1^k + \binom{k}{1}1^{k-1}(ax) + \binom{k}{2}1^{k-2}(ax)^2 + \ldots$$

$$\binom{k}{0} = \frac{k!}{(k-0)!\,0!} = \frac{k!}{k!\times 0!} = 1 \qquad \binom{k}{1} = \frac{k!}{(k-1)!\,1!} = \frac{k\times(k-1)!}{(k-1)!\,1!} = k$$

$$\binom{k}{2} = \frac{k!}{(k-2)!\,2!} = \frac{k(k-1)(k-2)!}{(k-2)!\times 2!} = \frac{k(k-1)}{2!} = \frac{k(k-1)}{2}$$

$$(1+ax)^k = 1 + k(ax) + \frac{k(k-1)}{2}(ax)^2 + \ldots \qquad \textbf{(1)}$$

$$(1+ax)^k = 1 + 24x + 264x^2 + \ldots \qquad \textbf{(2)}$$

Comparing Eq.1 and Eq.2:

$$ak = 24 \Rightarrow a = \frac{24}{k}$$

$$\frac{k(k-1)a^2}{2} = 264 \Rightarrow \frac{k(k-1)\left(\frac{24}{k}\right)^2}{2} = \frac{576(k^2-k)}{2k^2} = 264$$

$$\frac{576(k^2-k)}{2k^2} = 264 \Rightarrow 528k^2 = 576k^2 - 576k \Rightarrow -48k^2 = -576k \Rightarrow k = 12$$

and $a = \frac{24}{k} = \frac{24}{12} = 2$

$\backslash\ (1+ax)^k = (1+2x)^{12} = 1 + 24x + 264x^2 + \ldots$

Note: $(3-x)^{-1} = \left[3\left(1-\frac{x}{3}\right)\right]^{-1} = 3^{-1}\left[1+\left(-\frac{x}{3}\right)\right]^{-1}$ $\quad (3-x)^{-1} = \left[3\left(1-\frac{x}{3}\right)\right]^{-1} = 3^{-1}\left[1+\left(-\frac{x}{3}\right)\right]^{-1} = \frac{1}{3}\left[1^{-1} + (-1)\times 1^{-2}\left(-\frac{x}{3}\right) + \frac{(-2)(-1)}{2!}\times 1^{-3}\left(-\frac{x}{3}\right)^2 + \ldots\right]$

$= \frac{1}{3}\left[1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right] = \frac{1}{3} + \frac{x}{9} + \frac{x^2}{27} + \ldots$, Valid for $\left|\frac{x}{2}\right| < 1$

This can be verified by **long division** as follows: $\frac{1}{3-x}$

$$
\begin{array}{r|l}
 & \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \cdots \\
\hline
3 - x & 1 \\
 & -1 + \frac{1}{3}x \\
\hline
 & \frac{1}{3}x \\
 & -\frac{1}{3}x + \frac{1}{9}x^2 \\
\hline
 & \frac{1}{9}x^2 \\
 & -\frac{1}{9}x^2 + \frac{1}{9}x^3 \\
\hline
 & \frac{1}{9}x^3
\end{array}
$$

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